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I. *On the Electric Discharge through gases at very low pressures.* *By* Sir J. J. Thomson, *O.M., F.R.S.**

IN this paper I consider some properties of the discharge at low pressures, especially those which are associated with the neighbourhood of the cathode, such as the dark space and the distribution of potential through it, the origin of cathode and positive rays, and the formation of the negative glow. My experiments were made at pressures comparable with those occurring in X-ray tubes with cold cathodes.

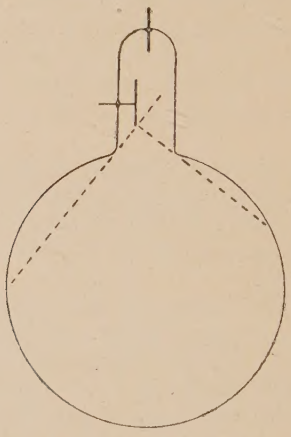
Let us first consider the sources of ionization in the dark space. I think the evidence is against this ionization being due, wholly or mainly, to the detachment of electrons by collisions of cathode rays or positive ions against the molecules of the gas.

We begin by considering the ionization due to the electrons driven from the cathode. Glasson's experiments (Phil. Mag. Oct. 1911) show that, as is indicated by theory, the number of ions produced by a cathode ray per unit length of path varies inversely as the kinetic energy of the ray. For rays moving with a velocity of $4{\cdot}7 \times 10^9$ cm./sec. through air at a pressure of 1 mm. of mercury, he found that 1·5 pairs of ions were produced by each ray in travelling over 1 cm. Under the usual conditions in gases at pressures where the dark space is well developed the velocity would be greater than this and the pressure much less than one

* Communicated by the Author.

millimetre, there would therefore be very little ionization by the electrons in passing over the dark space. Glasson (Proc. Roy. Soc. Tasmania, pp. 1–4, 1913) showed that the loss of energy by the electrons in their path is many times greater than can be accounted for by the work spent in ionization. This suggests that these high-speed electrons when passing through the molecules of a gas may displace some of the electrons in the inner layers of these molecules without ejecting them from the molecule. The return of these electrons to their old position would be accompanied by the emission of high frequency radiation which could ionize molecules on which it fell. Thus on this view the molecules traversed by the high-speed electrons emit radiation able to produce ionization. This radiation will spread throughout the surrounding gas, which will thus be exposed to ionizing radiation if it is within uninterrupted sight of a high-speed cathode ray. We suppose that the gas in this condition forms the negative glow, the luminosity of which is due to the recombination of the ions ; when this recombination is prevented by a strong electric field as in the dark space, the gas is not luminous.

Fig. 1.



A confirmation of this view is afforded by the fact, long ago pointed out by Goldstein, that the negative glow, a region where there is very intense ionization and where the electric force is very small, extends to parts of the tube not traversed by cathode rays. Thus in a tube like that in fig. 1, taken from Goldstein's paper, Wied. *Ann.* lxvii. p. 83 (1899), the negative glow extends to the region bounded by the

dotted lines. Thus, though the negative glow occurs in parts of the tube not traversed by cathode rays, it is confined to those parts which have an uninterrupted view of some part of the path of a cathode ray ; this is in accordance with the view that it is due to radiation from the molecules of gas traversed by cathode rays. It is not necessary that the parts of the tube where the cathode rays are developed should have an uninterrupted view of the place where the cathode ray strikes the walls of the tube, so that this ionizing radiation is not exclusively that produced by the impact of cathode rays against solid bodies.

The existence of radiation other than luminous and X-ray radiations has been known for a long time : thus E. Wiedemann (*Zeitschr. f. Elektrochemie*, p. 159, 1895) showed that rays which could excite thermoluminescence, "Entladungstrahlen," are given out by electric sparks; the properties of these rays have been investigated by Hoffmann (Wied. *Ann.* lx. p. 269, 1897) and Miss Laird (Phys. Review, xxix. p. 225 ; xxx. p. 293). The writer (Proc. Cambridge Phil. Soc. x. p. 74, 1899 ; xiv. p. 417, 1907 ; xv. p. 482, 1908) has shown that in the discharge at low pressures rays occur which have the power of ionizing gases, and that the intensity of emission of these rays varies from one part of the discharge to another. The radiations inside the tube which cannot pass through its walls are deserving of much more attention than they have received. One kind of such radiation which can easily be observed occurs at comparatively high pressures, say about 1 mm., when if the tube is made of soft German glass an olive-green phosphorescence, quite different from that due to cathode rays, appears on the walls of the tube when the discharge passes. I have shown that this radiation can be refracted so that it is probably so near the luminous spectrum that it can be regarded as a form of ultra-violet light.

Ionization produced by Positive Ions.

I do not think that the kind of ionization which results in the detachment of a free electron is often produced by the collisions of positive ions in the dark space. Such collisions cannot lead to ionization of this type unless the energy of the ions is exceedingly large, greater than some 70,000 volts if the ions are atoms of oxygen or nitrogen. As the energy must be proportional to the mass of the ions, for heavy ions it would far exceed that available with moderately sized coils.

There are two sets of considerations, one experimental the other theoretical, which lead to this conclusion.

The experimental evidence is, that the cathode rays after passing through the dark space all move at the same speed. Now, if ionization was going on in the dark space electrons would start from different parts of it and their kinetic energy when they were ejected from it would depend upon the place from which they started, and along with their kinetic energy their deflexions in electrostatic and magnetic fields would vary with their place of origin. Thus a narrow pencil of cathode rays formed by such electrons would spread out into a band when exposed to electric and magnetic forces. A similar pencil of cathode rays, if it consisted of electrons which had started from the cathode itself, would not spread out as all the electrons would have the same energy and suffer the same deflexion. Thus the study of the effect of electric and magnetic fields can give us a great deal of information about the ionization going on in the dark space, and I have made a large number of experiments on this subject.

The method adopted was as follows. A cathode with a plane face was placed in a side tube A, fused on to a large bulb; the cathode rays passed across the bulb and fell upon a metal plug with a hole bored through it placed at the junction with the bulb of another side tube B diametrically opposite to A. One of the very fine copper tubes used in positive ray experiments passed through the hole in the plug; the cathode rays leaving this tube formed a narrow pencil and passed through electric and magnetic fields of the type used to produce a positive ray spectrum. After passing through these fields the pencil fell on a Willemite screen at the end of the tube B, and in the absence of electric and magnetic forces produced a small circular patch of phosphorescence on the screen. When either the electric or magnetic field was in action the spot was deflected; it was not drawn out into a band, but remained a circular patch of the same size as the undeflected one with a very faint tail. This shows that the electrons which produce the phosphorescence are all moving at the same speed and that the effect on the screen of electrons moving at any other speed is negligible. By measuring the electrostatic deflexion of the rays we can deduce their kinetic energy, and can then compare it with the potential difference between the anode and cathode of the discharge-tube as determined by the equivalent spark-gap.

I have made experiments with different gases—air, hydrogen, oxygen, carbonic acid—and at different pressures,

and with the equivalent spark-gaps ranging from 2 to 35 millimetres; in all but a few exceptional cases to be considered later, the spot preserved its shape, it was not drawn out into a band, and the energy of the cathode rays as determined by their electrostatic deflexion was within the limits of the error inherent to the spark-gap method equal to that corresponding to the potential difference between the electrodes. In some cases when the "break" in the primary circuit of the induction-coil was working irregularly, there were several deflected spots to be seen; these were isolated and not connected by any band of luminosity, they were evidently due to changes in the potential differences between the terminals of the induction-coil due to the irregularities of the break.

The other reason for believing that ionization is not due to the collisions made by the positive particles follows from the consideration of the amount of energy that would be transferred to an electron by a collision with a positive ion. In consequence of the great mass of the ion in comparison with that of an electron, the energy of an ion if its speed were equal to that of an electron which possessed energy equal to the ionizing potential would be very much greater than that of the electron. Thus if the ionizing potential of the gas were 10 volts, the energy of a hydrogen molecule moving at the speed of an electron with the ionizing potential would be 36,000 volts, that of an atom of oxygen 288,000 volts, and so on. Thus, unless the velocities of the ions are much greater than those produced by ordinary induction-coils, the individual electrons in the ion possess far too little energy for them to be able to ionize as individuals.

Let us now consider the transference of energy from the ion to an electron, regarding the ion as a unit. If M_1, M_2 are the masses of two colliding particles, one of which may be regarded as at rest, T the kinetic energy of the moving one, the transfer of energy from the one particle to the other is

$$\frac{4M_1M_2}{(M_1+M_2)^2}T\sin^2\theta,$$

where 2θ is the angle through which the direction of the relative velocity of the particles is deflected by the collision; hence the maximum amount of energy which can

be transferred is

$$\frac{4M_1M_2}{(M_1+M_2)^2}T,$$

or when, as in the case we are considering, one of the masses M_1 is small compared with the other,

$$\frac{4M_1}{M_2}T.$$

Thus an atom of hydrogen could only transfer 4/1800 of its energy to an electron, so that if the ionizing potential were 10 volts it would have to have energy represented by 4500 volts to detach the electron; if the ion were an atom of oxygen its energy would have to be 72,000 volts. Thus the energy of all ions, except hydrogen and possibly helium, would have to be far greater than those available with the use of ordinary coils if they are to produce ionization by collisions. At very low pressures with large potential differences between the electrodes, it is possible that in hydrogen and helium we may have cathode rays with less than the normal energy. In a few cases I have observed that the pencil of cathode rays was spread out into a band by the electric and magnetic fields; this may have been due to hydrogen ions, which are nearly always present in the discharge through rarefied gases.

There is, however, another and very interesting way in which positive ions can produce ionization. When these ions reach the cathode they give up their charges to the electrode and become neutral. This neutralization is produced by an electron falling into the positive ion. Now, when an electron falls into an atom radiation is emitted; this radiation is done up into units, each unit possessing an amount of energy measured by the loss of potential energy due to the fall of the electron into the atom. When the electron falls into and neutralizes a positive ion, this energy is equal to the ionization potential of the neutralized ion, *i.e.* to the ionization potential of the atoms or molecules of the gas through which the discharge is passing. Thus, when a positive ion is neutralized, radiation is emitted which has enough energy when absorbed to ionize the atoms or molecules of any gas whose ionizing potential is ever so little less than that of the gas which supplied the ion. Thus at the cathode where the positive ions are neutralized, radiation is emitted which has all but enough energy to ionize the gas itself; this radiation will fall upon the metal of the cathode and give rise to photoelectric effects, as the energy per unit of

radiation necessary for these effects is less than that corresponding to the ionization potential of most gases. The electrons liberated by the photoelectric effects will start from close to the cathode, and thus when they emerge from the dark space will form a homogeneous system having energy represented by the full fall of potential in the dark space. The radiation which is not absorbed by the metal of the cathode will go out into the gas and will be able to ionize any molecule upon which it falls if the ionizing potential of the molecule is less than that of the ion.

Thus positive ions, though of insufficient energy to ionize by direct collision, may through the radiation they emit when they are neutralized produce ionization, which may extend over a volume large compared with that which could be reached if the ionization were due to collision.

It may be noted in this connexion that Horton and Davies (Proc. Roy. Soc. xcv. p. 333) have detected the emission of electrons from a metal plate struck by positive helium atoms with energy as low as 20 volts.

The sequence of events in the dark space is thus as follows :—

(1) The positive ions emit radiation when they give up their electrical charges at the cathode; this radiation we call radiation (i).

(2) Radiation (i) falling on the cathode causes a photoelectric emission of electrons from the electrode.

(3) These electrons acquire a high speed under the electric field, and then when traversing the molecules of the gas cause them to emit a radiation which we call radiation (ii).

(4) Radiation (ii) ionizes those portions of the gas which it can reach before it is absorbed.

(5) The recombination of the ions thus produced gives rise to the luminosity of the negative glow; this recombination consisting of the falling in of an electron to a positive ion will produce radiation which can ionize molecules whose ionizing potential is less than that of the neutral atom formed by the recombination. We shall call this radiation, radiation (iii).

Thus on this view there are several kinds of radiation near the cathode : there is radiation (i) starting from quite close to the cathode, (ii) starting from all regions traversed by the cathode rays, and (iii) coming from the negative glow.

The number of electrons ejected from the cathode by the action of radiation (i) may depend upon the energy possessed by the positive ions when they strike against the cathode as well as upon the number of such ions. For just as in the motion of positive rays through gases when a positive ion may get neutralized and then again positively charged by giving up an electron to a molecule against which it strikes, so a positive ion striking against the cathode may alternate from the charged to the uncharged condition, if it has much energy, many times before it loses its charge for the last time: each change from the positively charged to the uncharged state would be accompanied by the emission of radiation. The type of radiation (i) might be expected to depend mainly upon the nature of the positive ions; the number of ions ejected by the radiation would depend to some extent upon the kind of material used for the cathode.

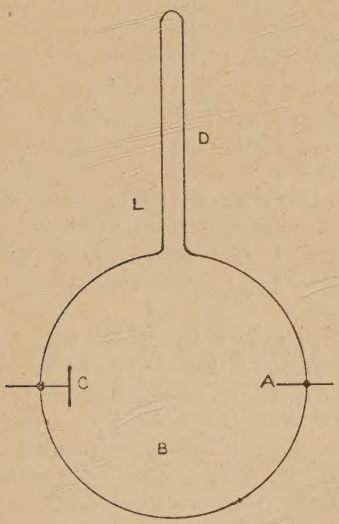
With regard to radiation (ii), the type of this radiation, *e. g.* its frequency and penetrating power, would depend on the energy the cathode rays possess when they excite it and also upon the nature of the molecules which are excited by the rays. As the energy of the cathode rays varies from point to point of their path, this radiation will not be homogeneous. The maximum energy of the cathode rays depends upon the current through the tube and the pressure of the gas, so that the properties of this kind of radiation will be a function of the pressure and of the current. The amount of radiation excited per cm. of path will be proportional to the pressure of the gas, and as the absorption of the radiation is also proportional to the pressure, if the cathode rays extend through a considerable volume of the gas the intensity of radiation (ii) at any point will not vary much with the pressure. As radiation (ii) is excited by cathode rays having considerable momentum part of which they lose when exciting this radiation, we should expect the momentum of the radiation to be in the same direction as that of the cathode rays, so that more of this radiation would travel forward in the direction of the cathode rays than would travel back.

The type of radiation (iii) which arises from the recombination of the ions produced by radiation (ii) will depend mainly upon the character of the gas in the tube. It is of greater wave-length than (ii), not extending much below 1000 Ångström units.

Nature of Radiation (ii).

The following experiment throws, I think, some light on the nature of this radiation. B (fig. 2) is a large bulb with a cathode at C and an anode at A; D is a long narrow tube about 50 cm. long. Within a certain range of pressure the negative glow extends right up to the top of D. A strong electromagnet was placed at L: when this magnet was in action the negative glow in D was not affected except in the region close to L; here it was extinguished, and the glow

Fig. 2.



formed two luminous columns separated by a short non-luminous layer. The ends of the luminous column showed much the same relation to the magnetic force acting upon them as does the luminous boundary of the dark space when acted on by magnetic force. Though the magnetic field produced this local disturbance, the upper part of D remained filled with glow whose brightness did not seem much affected by the magnetic field. This experiment shows, I think, that radiation (ii) does not in the main consist of secondary cathode rays, for if it did the rays could not have crossed the magnetic field at L unless their velocity had been considerably greater than the primary cathode rays. I conclude then that radiation (ii) is in the main radiation of the character of light or Röntgen rays, and not a corpuscular radiation.

Production of Positive Ions in the Dark Space.

Positive ray photographs show that a narrow pencil of positively charged atoms or molecules of one kind, *e. g.* atoms or molecules of hydrogen, is drawn out into a continuous band when the rays are acted on by electric or magnetic forces; showing that there are some atoms whose energy is equal to E where E may have any value between certain wide limits. This is in contrast to the behaviour of the cathode rays, for we have seen that except in very special circumstances practically all the high-speed electrons possess the same amount of energy. It is natural to attribute the variation in energy of the positive ions to their having been produced at different parts of the dark space and having, therefore, on their arrival at the cathode fallen through varying potential differences. If, however, the process which produces the positive ion was the dissociation of an atom or molecule into an electron and positive part, the electrons on leaving the dark space would also have fallen through varying potential differences, would possess different energies, and a pencil of cathode, like one of positive rays, would be drawn out into a continuous band by electric and magnetic fields.

There are two processes by which we could get charged positive ions without electrons. The first is by the impact of a positive ion on a diatomic molecule, of which one atom may be regarded as positively, the other as negatively electrified. Now, though the collision of a positive ion with a molecule could not, unless the energy of the ion were exceedingly large, separate an electron from the molecule, it might, if the molecule consisted of two oppositely charged ions, separate these ions, and dissociate the molecule into a positive and a negative ion. It could do this because the masses of these ions are much more nearly equal to its own mass than is the mass of an electron, so that a much greater fraction of the energy of the positive ray could be communicated to the ion than to the electron. Thus, while the fraction given to the electron would be too small to detach it, the fraction transferred to the ion might be large enough to tear it away. The fact, however, that the variation of the energy of the positive rays occurs in monatomic gases like helium and argon as well as in diatomic ones, shows that this cannot be the only way in which the variation of the energy of the positive rays can be produced. Though a positive ray might not be able to detach and set free an electron from a molecule with which it came in contact, it

might in virtue of its charge be able to capture an electron from a molecule against which it struck and carry it away. The electron would neutralize the positive ray, and leave the molecule or atom from which it had been captured positively electrified. The positive ray would not communicate its energy to this atom or molecule, so that this would start on its career as a positive ray with little or no energy; the potential difference it would have to pass through before it reached the cathode would be less than the full fall of potential across the dark space, so that its energy when it reached the cathode would be less than the maximum.

The neutralized atom would continue to move with a high velocity, and if it came into contact with a molecule of the gas through which it is passing might either carry an electron away or leave one behind; in either case a pair of ions would be produced. The repetition of this process might give rise to a considerable number of positive and negative ions, but not to any electrons.

We can obtain an integral equation to determine the potential in the dark space under such conditions. Let $\phi(x)$ be the potential at a distance x from the luminous boundary of the dark space. Let λ be the average distance traversed by an ion before transferring its charge to another. Then the chance of this transference taking place between ξ and $\xi+d\xi$ is

$$\epsilon^{-\frac{\xi}{\lambda}}\frac{d\xi}{\lambda},$$

and the chance that the ion passes through a distance x without transferring its charge is $\epsilon^{-\frac{x}{\lambda}}$.

If a positive ion starts from rest at a point ξ, its velocity when it reaches x will, if there is no loss of energy, be equal to

$$\sqrt{(2e/\mathrm{M})\{(\phi(x)-\phi(\xi)\}},$$

where M is the mass of the ion. Hence if i_0 is the number of ions starting from near the luminous border of the dark space per second and therefore $i_0\epsilon^{-\frac{\xi}{\lambda}}.d\xi/\lambda$ the number which start from the layer between $\xi+d\xi$ and ξ, the density of positive electricity at x will, if we neglect the effect of ions which have experienced more than one transference of charge, be

$$\frac{ei_0\epsilon^{-\frac{x}{\lambda}}}{\sqrt{(2e/\mathrm{M})\phi(x)}}+ei_0\int_0^x\frac{\epsilon^{-\frac{\xi}{\lambda}}d\xi}{\lambda\sqrt{(2e/\mathrm{M})(\phi(x)-\phi(\xi))}}.$$

The density of electricity is $\frac{1}{4\pi}\frac{d^2\phi}{dx^2}$, and since the density of the negative electricity is small compared with that of the positive, we may equate these expressions and get an integral equation for $\phi(x)$; this I have not been able to solve except when $\lambda=\infty$, when $\phi(x)=(9\pi i_0 e)^{2/3}x^{4/3}$.

We can see, however, that since the transference we are considering diminishes the average velocity of the ions when they approach the cathode, it will increase the density of the positive electricity near the cathode and thus make the force increase more rapidly in that region than the distribution of potential represented by $x^{4/3}$, the law when each ion reaches the cathode without interruption.

If an appreciable fraction of the current is carried by positive ions, then at low pressures and with fairly large currents all the molecules of the gas in the discharge-tube may have to strike against the cathode many times per second. Thus if the pressure is ·01 mm. of mercury and the current through the tube carried by the positive ions is one milliampere, in one second 10^{-4} electromagnetic unit of electricity or 3×10^6 electrostatic units must be given up at the cathode. As the charge on an ion is $4{\cdot}77\times10^{-10}$ electrostatic units, a charge of 3×10^6 units requires $6{\cdot}2\times10^{15}$ ions. In one centimetre of gas at ·01 mm. at 0° C. there are $2{\cdot}75\times10^{19}/760\times10^2$, or $\cdot36\times10^{15}$ molecules ; hence in one second the molecules contained in about 17 c.c. of the bulb must come in contact with the cathode. Unless this flow of ions to the cathode is balanced by a backward flow of molecules, the pressure at the cathode end of the tube must increase continually.

To produce this backward flow the pressure must increase as we approach the cathode. Taking the case of a plane cathode at right angles to the axis of x, then if n is the number of molecules per c.c. at a distance x from the cathode, the flow of molecules from the cathode due to the pressure gradient is per unit area per second

$$-\frac{dn}{dx}\cdot \mathrm{D},$$

where D is approximately $\frac{1}{3}\lambda c$, where λ is the free path of the molecule and c the average velocity of translation of the molecules. If this flow is to balance that of positive ions towards the cathode, then if i is the current per square centimetre carried by the positive ions,

$$\frac{i}{e}=\frac{c}{3}\lambda\frac{dn}{dx}.$$

In consequence of the large value of D at low pressures a very small pressure gradient is sufficient to produce a flow which will balance the flow of gas caused by the motion of the ions. Thus to take the numerical case just considered where $i/e = 6{\cdot}2 \times 10^{15}$, suppose that c equals $4{\cdot}6 \times 10^4$, the value for a molecule of oxygen at 0° C. Then

$$1{\cdot}5 \times 10^4 \lambda \frac{dn}{dx} = 6{\cdot}2 \times 10^{15},$$

or

$$\lambda \frac{dn}{dx} = 4{\cdot}1 \times 10^{11}.$$

At a pressure of ·01 mm. n will be of the order of $3{\cdot}5 \times 10^{14}$, so that

$$\lambda \frac{dn}{dx} = \frac{n}{850} \quad \text{approximately.}$$

Thus a gradient in the number of molecules which only amounted to a change of one part in about 850 in a length equal to the free path would be sufficient to counterbalance the pressure effects of the flow of positive ions to the cathode.

On the view that the main supply of the positive ions comes from near the luminous boundary of the dark length, we should expect that the backward flow of the uncharged molecules would be confined to the dark space, as such a flow would be sufficient to keep things in a steady state. The backward flow in addition to uncharged molecules may contain (see 'Rays of Positive Electricity,' p. 134) negatively charged molecules, and "retrograde" positively charged ones, though the number of these is small in comparison with the flow of positive ions to the cathode or of uncharged molecules away from it.

It is important to consider whether the values of current and potential observed when discharges pass through gases at low pressures are compatible with the view that the potential difference is that calculated from the distribution of positive electricity through the dark space. The lines on which the potential can be calculated are those given on p. 11, and although we cannot get an exact solution for the potential in that case, it is easy to see that the order of magnitude of the potential will be the same as that for the effect of a space-charge when all the electrons come from one electrode. In this case

$$i = \frac{\sqrt{2}}{9\pi} \sqrt{\frac{e}{M}} \frac{V^{3/2}}{x^2}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where i is the current per unit area of the electrode, V the

potential difference, and x the distance between the electrodes; M the mass of the carrier of the electric charge. All the quantities are measured in electrostatic units.

We shall in applying this formula to the discharge through rarefied gases take x as the thickness of the dark space and V the potential difference across that space; i will be the density of the current carried by the positive ions, not the total current through the tube. Let us take the case when V is 30,000 volts, in electrostatic measure 100, and when the thickness of the dark space is 6 cm. If the carriers of the positive electricity are atoms of hydrogen $e/M = 3 \times 10^{14}$; hence by (1) $i = 10^8/4$ in electrostatic units or 1/120 ampere, *i. e.* about 8 milliamperes. If the current were carried by atoms of nitrogen instead of hydrogen, $i = \cdot 57$ milliampere. These values of i are high considering that i is only the part of the current carried by the positive ions and is probably but a small fraction of the total current. At higher pressures, when the discharge passes more easily, the values of i are much smaller and do not seem necessarily incompatible with the magnitude of the currents flowing through the tube.

It must be remembered that the interpretation of such an equation as that we have just quoted depends upon the answer to two questions, neither of which has been definitely settled:

(1) Is the discharge at these low pressures continuous even when a constant difference of potential is applied to the electrodes?

(2) Is there an abrupt change in potential at the cathode?

With regard to (1) it is evident that if the discharge is intermittent the real value of i when the discharge is passing may be very much greater than the average value recorded by an ammeter. Dauvillier has found that the discharge through an X-ray bulb is intermittent even when a constant potential difference is used to drive the bulb. From general considerations we might, I think, expect variations in the current even under constant potential. For even when a gas is in a steady state, if absolutely instantaneous photographs could be taken of its molecules, there would be no resemblance between the photographs taken at different times; on the other hand, if the time of exposure were considerable in comparison with the time of relaxation, there

would be but little difference between successive photographs. Thus we see that it is only for effects which extend over a time longer than the time of "relaxation" of the gas that we can regard the gas as being in a "steady" state; for shorter times both the gas and the effects must be regarded as variable. In the discharge through gases at low pressures, the time it takes for an electron or charged atom to cross the dark space may be but a very small fraction of the time of "relaxation" of the gas; the gas cannot be regarded as in a "constant" state for these electrical effects, and the variation in the state of the gas may produce variations in the intensity of the current under a constant potential difference.

The evidence with regard to (2) is contradictory at moderate pressures: Westphal finding that there is a considerable drop of potential close to the cathode, while Aston finds no trace of any abrupt change in the potential. If such a drop exists, then equations of the type of (1) connecting potential and current cannot be applied, until we know how much of the potential is used up close to the cathode.

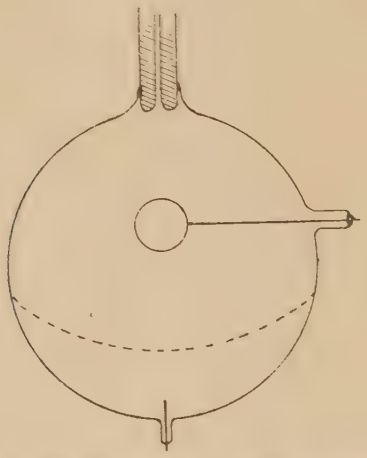
Another important point on which we have at present no direct information is the proportion between the current carried by the positive ions and that carried by the electrons. The fact that in X-ray bulbs the heating effect at the anticathode—the place bombarded by electrons—is very much greater than that at the cathode itself—the place bombarded by the positive ions—suggests that much the greater part of the current is carried by electrons. We shall see that this view is helpful in explaining many of the properties of the discharge at low pressures.

The question arises as to where the positive ions are produced. Are they produced in the positive column and drift into the dark space where they get accelerated, or are they produced in the dark space itself? The following experiment shows, I think, that a considerable quantity at any rate are produced in the dark space itself. In the tube used to take positive-ray photographs a quartz bulb was placed in front of the cathode at about the middle of the dark space (fig. 3); if the positive ions all drifted in from the cathode they would be stopped by the quartz bulb and no positive-ray photographs would be produced, as a matter of fact excellent photographs were taken with the quartz bulb in this position.

We shall suppose that ionization takes place in the dark space but that the number of ions produced in this region is much less than the number of electrons which start from the cathode; thus by far the greater number of electrons which

emerge from the dark space will possess the maximum energy corresponding to the fall through the whole potential difference in the dark space, while the energy of the positive ions which strike the cathode will vary with their place of origin in the dark space.

Fig. 3.



We shall now proceed to apply these ideas to some of the properties of the discharge at low pressures; we shall begin with the formation of the dark space.

We have supposed that the ionization in this space is due to the radiations (i). (ii). (iii). Let q be the intensity of the radiation and ρ the density of the gas; the rate of ionization per unit volume will be $k\rho q$, where k is a quantity which does not depend upon ρ or q. If the intensity of the radiation is constant throughout the dark space, and if d is the thickness of that space, the rate of ionization in the dark space estimated per square centimetre of the cathode is $kq\rho d$, and since there is no recombination in the dark space this will be the number of positive ions which strike against the unit area of the cathode per second. The emission of electrons from the cathode is on our view due to the neutralization of the positive charges: it will be proportional to the number of such charges and to their energy when they reach the cathode: the average energy will be proportional to the cathode fall of potential V. Thus if γ is the number of electrons emitted per second per unit area of cathode.

$$\gamma = Ak\rho q d\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A may depend upon the nature of the ions and the material of which the cathode is made.

We have supposed the ionization to be due to radiation primarily excited by the passage of the cathode rays through the gas ; the intensity q of this radiation will therefore be proportional to γ. The energy communicated to a molecule by an electron whose energy is V is proportional to

$$\frac{a\mathrm{V}}{c+b\mathrm{V}^2}.$$

The energy communicated to the molecules per unit path will also be proportional to ρ the density of the gas ; on the other hand, the absorption of the radiation by the gas will also be proportional to ρ, so that if we are dealing with the radiation from a large volume of gas the intensity of the radiation reaching any point will not vary much with ρ. Hence we may put

$$q = \mathrm{B}\gamma . \frac{a\mathrm{V}}{c+b\mathrm{V}^2},$$

where B is a quantity which in general does not vary rapidly with ρ but may do so under special conditions. Substituting this value of q in (2) we get

$$1 = \mathrm{BA}k\rho d . \frac{a\mathrm{V}^2}{c+b\mathrm{V}^2},$$

$$\text{or} \qquad \rho d = \frac{1}{\mathrm{BA}k}\left(\frac{b}{a}+\frac{c}{a\mathrm{V}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus when B is independent of ρ, the thickness of the dark space is given by an equation of the form

$$\rho d = \mathrm{C}+\frac{\mathrm{D}}{\mathrm{V}^2},$$

where C and D are constants if the tube and the nature of the gas with which it is filled remain unaltered.

The distribution of potential in the dark space may be calculated as follows, provided the potential depends only upon the distance from the cathode :—

Let $\phi(x)$ be the potential difference between the luminous boundary of the dark space and a place P at a distance x from it. The positive charge at P will have travelled to it from places like Q, whose coordinates ξ are less than x. If a positive ion started from Q and fell freely to P its velocity at P would be equal to

$$\sqrt{\frac{2e}{\mathrm{M}}(\phi(x)-\phi(\xi))}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where M is the mass of the positive ion. If r is written for

$kq\rho$, the number of positive ions which start per second from a layer of thickness $d\xi$ at Q is $rd\xi$. The density of these when they reach P will be $rd\xi/v$, where v is written for the expression (4); hence the density of the positive charge at P is

$$e\int_0^x \frac{rd\xi}{v}.$$

As the negative electricity is carried by electrons which move much more quickly than the positive ions, the density of the negative electrification will be negligible in comparison with that of the positive. Hence the resultant density of the electrification will be practically that of the positive charges, so that

$$\frac{1}{4\pi}\frac{d^2\phi}{dx^2} = e\int_0^x \frac{rd\xi}{v} = r\sqrt{\frac{Me}{2}}\int_0^x \frac{d\xi}{\sqrt{\phi(x)-\phi(\xi)}}, \quad . \quad (5)$$

an integral equation to find $\phi(x)$. To solve it put $\phi(x) = Ax^n$, when equation (5) becomes

$$\frac{A}{4\pi}n(n-1)x^{n-2} = r\sqrt{\frac{Me}{2A}}\int_0^x \frac{d\xi}{\sqrt{x^n\left(1-\left(\frac{\xi}{x}\right)^n\right)}}$$

$$= r\sqrt{\frac{Me}{2A}}\frac{1}{x^{\frac{n}{2}-1}}\int_0^1 \frac{dz}{\sqrt{1-z^n}}.$$

Thus

$$n-2 = -\left(\frac{n}{2}-1\right)$$

or $$n = 2,$$

and $$A^{3/2} = r\pi^2\sqrt{\frac{Me}{2}},$$

so that

$$\phi(x) = (r\pi^2\sqrt{Me/2})^{2/3}x^2; \quad . \quad . \quad . \quad . \quad (6)$$

so that if d is the thickness of the dark space and V the potential difference across it,

$$V = (r\pi^2\sqrt{Me/2})^{2/3}d^2.$$

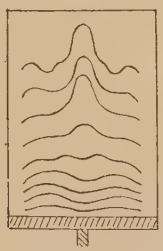
If I is the current carried by the positive ions $erd = I$, hence

$$V = \pi^{4/3}\left(\frac{M}{2e}\right)^{1/3}I^{2/3}d^{4/3}.$$

The distribution of potential represented by equation (6) is that found by Aston (Proc. Roy. Soc. lxxix. p. 526) ; his experiments were, however, made at comparatively high pressures and with small potential differences.

The equipotential surfaces corresponding to this solution are planes parallel to the cathode. Wehnelt has shown however, that in consequence of the electrification on the walls of the tube the discharge tends, even with large cathodes, to concentrate at the centre of the cathode and the equipotential surfaces are distorted into the shapes represented in fig. 4.

Fig. 4.



We see from the expression for V that when the potential across the dark space is given, the thickness of the dark space varies inversely as the square root of the current.

The following calculation illustrates the effect of pressure and of the volume occupied by the negative glow on the value of q the intensity in the dark space of the ionizing radiation. Suppose that a unit of this radiation is liberated at P by a cathode ray, let λ be the mean free path of this unit, *i. e.* λ is the average distance the unit travels before it ionizes a molecule. Let δv be an element of volume at a point Q; then if $\delta\omega$ be the solid angle subtended by δv at P, the chance that the unit of radiation will start in the right direction to reach the element of volume at Q is $\delta\omega/4\pi$. If r is the distance of Q from P, the chance that the unit of radiation will ionize a molecule while going through the distance dr near Q is $e^{-\frac{r}{\lambda}}\frac{dr}{\lambda}$; hence the chance that the ray from P ionizes in the volume δv is

$$\frac{\delta\omega}{4\pi}e^{-\frac{r}{\lambda}}\frac{dr}{\lambda},$$

or, since $r^2\delta\omega\delta r=\delta v$,

$$\frac{e^{-\frac{r}{\lambda}}.dv}{4\pi r^2\lambda}.$$

The number of ions produced per unit volume in the dark space by a single unit of radiation is thus $e^{-\frac{r}{\lambda}}/4\pi r^2\lambda$. If these rays are produced uniformly throughout the negative glow, and if n_0 is the number of such rays starting from unit volume per second, the ionization per second per unit volume at Q will be

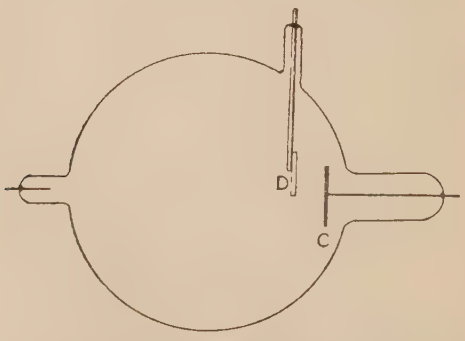
$$\frac{n_0}{4\pi}\iiint\frac{e^{-\frac{r}{\lambda}}}{r^2\lambda}\cdot r^2dr\sin\theta d\theta d\phi,$$

where the integration is extended over the region traversed by the cathode rays. Performing the integration we find for the ionization the value

$$\frac{n_0}{4\pi}\left(1-e^{-\frac{L}{\lambda}}\right)\Omega,$$

where Ω is the solid angle subtended at Q by the region traversed by the cathode rays, and L the value of r at the outer boundary of this region. This is the quantity we denoted by $kq\rho$; n_0 the number of units of radiation produced per unit volume by the cathode rays will be proportional to ρ, thus q will be proportional to $\left(1-e^{-\frac{L}{\lambda}}\right)$; if L/λ is considerable q will be independent of both L and λ and hence of ρ, and hence by equation (3), ρd will be independent of ρ except in so far as c/aV^2 involves ρ. If however L/λ is small, q will be proportional to L/λ ; and since $1/\lambda$ varies as ρ, q will be proportional to ρ, and $\rho^2 d$ and not ρd will be constant. A case in which the length of the dark space varies with the pressure more rapidly than $1/\rho$ is that represented in fig. 5.

Fig. 5.



C is a plane cathode from which cathode rays travel horizontally and strike against an insulated piece of plate glass D.

The bulb is for the most part filled with negative glow, but in addition to the usual dark space in front of the cathode there is another dark space in front of D not in general of the same thickness as that in front of C.

The negatively charged cathode rays starting from C carry a definite current into the plate D ; as this plate is insulated, when things have settled down into a steady state, an equal current must flow back from D across the dark space in front of it. As this space is dark there is no recombination of the ions within it, so that the current through it will be saturated and measured by the number of ions produced in this space per unit time ; it is also equal to the current carried by the cathode rays which strike D, so that the ionization in this dark space, like that in front of C, will be directly proportional to the number of cathode rays coming from unit area of C. The region which can send ionizing radiation to the dark space in front of D is much smaller than that which can send such radiation to the dark space at C, so that while the thickness of the dark space at C should vary as $1/\rho$, that at D should vary more rapidly with the pressure.

I made a series of measurements of the relative thicknesses of the primary dark space at C and the secondary one at D at different pressures; the gas in the bulb was oxygen, because the negative glow is denser and the outline of the dark space more sharply defined than in any other gas I have tried. The results are given in the following table.

Primary dark space at C.	Secondary dark space at D.
20 mm.	1·5 mm.
25 mm.	2·5 mm.
28 mm.	3·5 mm.
35 mm.	5·0 mm.
45 mm.	7·0 mm.

It will be seen from these numbers that as the thickness of the dark space at C increases as the pressure diminishes, that at D increases still more rapidly. If, as many experiments indicated, the thickness of the cathode dark space varies as $1/\rho$, that of the dark space at G must vary as a higher power of $1/\rho$. The measurements are consistent with this thickness varying as $(1/\rho)^2$.

The Negative Glow.

We shall consider in more detail the properties of the negative glow. This luminous glow forms the outer boundary of the dark space, its appearance and the volume through which it extends depend upon many factors such as the pressure of the gas, the current through the tube, the shape of the tube, the paths of the cathode rays through it, and the nature of the gas in the tube. In the negative glow the electric force is exceedingly small in comparison with its value in other parts of the tube, indicating that there is very intense ionization in the glow. This ionization, on the view we have suggested, is produced primarily by the radiation (ii) given out by molecules of gas traversed by cathode rays; the ions and electrons produced in this way recombine, and in doing so give out luminous radiation and a radiation corresponding in energy to the ionizing potential of the system formed by the electron and positive ion after recombination. Thus in a mixture of ions of different ionizing potentials, the recombination of those with a high ionizing potential will give out radiation capable of producing a fresh supply of the ions with a smaller ionizing potential. The ionization in the negative glow is not merely the detachment of an electron from a molecule: if this were so, all the positive ions would be molecules; the evidence of the positive rays shows that these molecules are often broken up into atoms and that these atoms may combine and form new compounds. Thus, though only one kind of compound gas be put in the discharge-tube, we find on analysis by positive rays that there may be twenty or thirty different kinds of carriers of positive electricity due to new arrangement of the atoms which were in the molecules of the compound gas.

The conditions in the negative glow seem to be those of thermodynamic equilibrium, *i. e.* if the gas filling the tube is a compound containing say oxygen, hydrogen, and carbon atoms in certain proportions, then in the negative glow there will be all possible compounds of these elements, and these compounds will be in the proportions corresponding to thermodynamic equilibrium at a suitable temperature.

We know that in many cases the elements are able to exist in stable equilibrium in proportions which bear no relations to those indicated by thermodynamics. Thus, mixtures of hydrogen and oxygen in the proportion of two volumes of hydrogen to one of oxygen will exist side by side at room temperatures without the occurrence of any chemical combination, whereas thermodynamical considerations would

indicate that practically all the atoms should be united in H_2O.

The reason that thermodynamic equilibrium is not attained under ordinary conditions is, to take the example just considered, that before H_2O can be formed from a mixture of H_2 and O_2, work has to be expended to split the oxygen molecules into atoms. Although the final result of the formation of water from hydrogen and oxygen results in a great liberation of energy, yet, before this result can be obtained, there must be a preliminary expenditure of energy and there must be some sources from which this energy can be borrowed. To lend this energy is the function of a catalyst, and anything which can supply the energy sufficient to enable the reaction to start on a small scale will act as a catalyst. Now, in the negative glow we have many sources of energy which are not present in gases in the normal state. We have energy stored up in radiations of various kinds, we have the energy of moving electrons : all these will act as catalysts and enable the gases in the negative glow to be in thermodynamic equilibrium. I have made a good many experiments to test whether or not this is so. The method was as follows : I filled the discharge-tube with, say, carbon monoxide and obtained the positive-ray spectrum, then I pumped out the CO and filled the tube with CO_2, and again obtained the positive-ray spectrum. There were differences between the spectra of CO and CO_2, but these were not so much in the absence or presence of lines corresponding to different atoms or molecules, for as a matter of fact if a line is present in one of these spectra, traces at least of it are found in the other ; the differences in the two spectra are in the relative intensities of the lines ; this, however, might be because the proportion between the number of carbon atoms and of oxygen ones is not the same in CO as in CO_2. When, however, oxygen is mixed with CO in the proportion of one volume of oxygen to two of CO, so that the mixture is represented by $2CO + O_2$, the proportion between the carbon and oxygen atoms is the same as for CO_2. On comparing the positive-ray photograph of the mixture with that of CO_2, I could not detect any differences between them. I made similar experiments with formaldehyde COH_2 and the mixture $CO + H_2$, with methyl alcohol $CH_3 . OH$ and the mixture $CO + 2H_2$, with formic acid $CHO . OH$ and the mixture $CO_2 + H_2$, and found that in each of these cases the positive-ray spectrum of the mixture was the same as that of the compound. This is the result we should get if

the gas in the negative glow were in thermodynamic equilibrium at a high temperature, for the composition of a mixture of carbon, oxygen, and hydrogen atoms in thermodynamic equilibrium depends at a given temperature only upon the number of atoms of carbon, oxygen, and hydrogen, and not upon their state of combination when introduced into the discharge-tube.

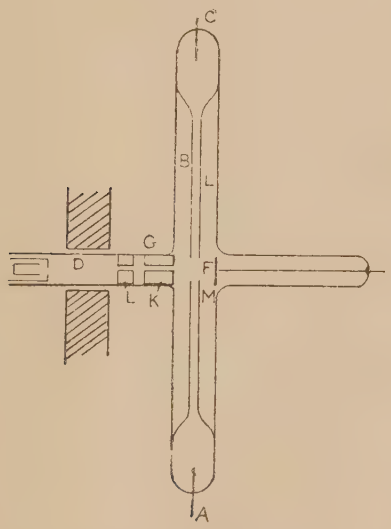
Incidentally we may mention that the positive-ray photographs of carbon compounds, and especially of such compounds as benzene, phenol, and hydrocyanic acid, show that in the negative glow carbon has a great tendency to polymerize, for in the positive-ray photographs of these gases we find lines corresponding to C_1, C_2, C_3, C_4, C_5.

It takes only a very short time for the gas in the discharge-tube to settle down into thermodynamic equilibrium when the electric discharge is passing through it. This is shown by the fact that the lines corresponding to the different atoms or molecules appear on photographs taken with the "instantaneous" discharge. The "instantaneous" discharge is that produced when the induction-coil which produces the discharge is not allowed to run continuously. Instead of using the ordinary interrupter to break the primary circuit, the primary circuit is broken by hand, using a break with a very strong spring so as to make the interruption of the circuit as abrupt as possible. I found that a spring gave as abrupt a break as Lord Rayleigh's method of breaking the primary circuit by a bullet fired from a revolver, and was much more convenient. I was not able to get the discharge sufficiently intense to photograph a single flash, but what was equivalent to a photograph of a single flash was obtained by passing a steady stream of the gas under investigation through the discharge-tube and making the interval between the flashes so long that all the gas that was in the bulb when one flash was taken was washed out before the next flash. I got distinct photographs with twenty or thirty flashes, and these showed lines corresponding to a great variety of atoms and molecules, showing that these were produced in the time taken by the flash. I estimated from the results given by Professor Taylor Jones for a coil of somewhat similar dimensions that the duration of each discharge was about 1/3000 sec. Each of these flash discharges shows all the usual characteristics of the discharge through gases at low pressure, *e. g.*, the dark space, the negative glow, and the positive column.

The Ions in the Positive Column.

In the part of the luminous discharge which is in contact with the anode, the electric field is very much weaker than it is in the dark space ; and this part of the discharge can be shielded easily from the radiations which we have supposed to produce the ionization in the negative glow and the dark space. The ionization in the positive column we regard as due to ionization by the collisions of electrons : it is therefore a matter of some interest to investigate the nature of the ions in the positive column to see whether they approach in variety and complexity those which pass through the dark space and are revealed by positive ray analysis. I have made experiments on this point, and though the research is not yet complete evidence has been obtained that the character of the ions in the positive column is similar to that of those in the positive rays, though we are not yet in a position to say that the ions are the same in all respects.

Fig. 6.



The method used to investigate the ions in the positive column is that indicated in fig. 6. The current from an induction-coil passed from the cathode C to the anode A, through two glass tubes L and M separated by an interval and adjusted so that the axes of the tubes were in the same straight line. The positive column filled the greater part of

these tubes and stretched across the interruption between them. The nature of the ions in the interval between the tubes was investigated in the following manner. The glass tube G carries a tightly-fitting metal plug K with a hole bored through it, a long fine copper tube of the kind used in my experiments on Positive Rays fitted tightly into this hole. One end of this tube came opposite the interval between the tubes L and M. This fine tube was the only channel of communication between the vessels B and D, so that by means of powerful pumps and charcoal cooled with liquid air a good vacuum could be maintained in D, even when the pressure in D was comparable with that due to a millimetre of mercury.

To drive the positive ions from the positive column through the narrow tube in the plug, an electric field was maintained between the side electrode F and the plug K. After passing through the fine tube the ions were accelerated before passing through a hole in another plug L. After going through L they pass between the poles of an electromagnet and then pass through a circular hole in a closed brass vessel, behind this hole is a Faraday cylinder connected with a Dolezalek electrometer. The potential differences required to drive the ions through the tube and accelerate them afterwards were derived from a high potential dynamo made by Evershed and Vignolles which gave steady potential differences up to 6000 volts. The current from this dynamo ran through two high resistances—two tubes A and B filled with a mixture of xylol and alcohol. A was inserted between F and K, B between K and L. The resistance of A is small compared with that of B, so that the energy which the ions possess when they pass through the magnetic field is very nearly that due to the potential difference between the terminals of the dynamo, and is the same for all ions. The deflexion by the magnetic field of an ion whose mass is m and velocity v is equal to

$$\frac{e}{mv}\mathrm{B},$$

where B is a quantity which depends upon the strength of the magnetic field and upon the length of path traversed by the ion before it reaches the aperture in front of the Faraday cylinder. B was determined by the method given in my 'Rays of Positive Electricity,' 2nd edition, p. 18. If V is the potential difference between the terminals of the dynamo,

then approximately

$$\tfrac{1}{2}mv^2 = \mathrm{V}e\,;$$

hence the deflexion of the ion is equal to

$$\left(\frac{e}{m}\right)^{1/2}\frac{\mathrm{B}}{\sqrt{2\mathrm{V}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

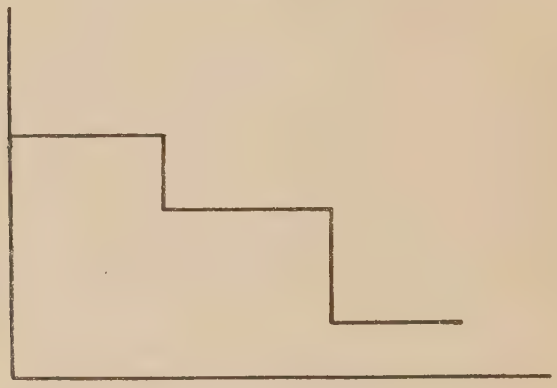
Let us now consider what is the effect of gradually increasing the magnetic field on the charge of positive electricity received by the Faraday cylinder and measured by the electrometer. The apparatus is adjusted so that when there is no magnetic field the ions pass through the centre of the aperture in front of the Faraday cylinder. No effect on the charge received by the cylinder will be produced until the deflexion of the most deflected ion (the atom of hydrogen) is equal to the radius of the circular aperture in front of the Faraday cylinder.

When the magnetic force reaches a value just large enough to produce this deflexion, the charges which were carried to the Faraday cylinder by the atoms of hydrogen are stopped by the walls of the vessel round it, and the deflexion of the electrometer falls—abruptly if the pencil of rays is very narrow, gradually if it is diffuse. After this diminution in the deflexion of the electrometer there will be no further diminution until the magnetic force reaches the value for which the deflexion of the ion next in mass to the hydrogen atom, *i.e.*, the hydrogen molecule, is equal to the radius of the aperture ; at this stage there will again be a sudden diminution in the deflexion of the electrometer. Thus we see that if we plot the deflexion of the electrometer as ordinate against the magnetic force as abscissa, we should with a very narrow beam of positive ions get a graph like fig. 7.

Each nick in the graph indicates the presence of an ion, which under the magnetic force represented by its abscissa is deflected through a distance equal to the radius of the aperture. Knowing this radius and the magnetic force, we can from the expression (7) determine the value of m for each of the ions. In my experiments the beam of positive ions was widened by the scattering it experienced in its passage up to the Faraday cylinder, so that the nicks in the graph were rounded off and it was not possible to obtain more than approximate values of m. They were, however, sufficiently well marked to determine with ease whether or not a particular ion was to be found among those drawn

from the positive column. I have found in this way that in the positive column in air there are, even when the pressure approaches 1 mm. of mercury, atoms and molecules of hydrogen, atoms of either oxygen or nitrogen or both—the nicks due to these are too near together to be separated with certainty in my curves—as well as molecules of these elements. These experiments show that the ionization in the

Fig. 7.



positive column involves the splitting up of molecules into atoms as well as the abstraction of electrons from molecules. This suggests that in the positive column as in the negative glow we have a distribution of atoms and molecules analogous to that which represents thermodynamic equilibrium of the gases in the tube at a high temperature.

It is remarkable that these experiments show no traces of negative ions, as distinct from electrons, in the positive column; when the electrode F is made negative, instead of positive, so as to drive negative instead of positive ions into the Faraday cylinder, no appreciable deflexion of the electrometer occurs; though when the potential of F is reversed so as to drive positive ions into the cylinder, the electrometer may show a deflexion of more than 100 divisions in a minute.

Theory of the Uniform Positive Column.

We suppose that the ionization in the positive column is due, (1) to collisions with free electrons, (2) to radiation given out when a positive ion is neutralized by an electron falling into it. Positive and negative *ions* are supposed not to produce ions in the positive column by collision.

Let n_1, n_2, m be respectively the number of free electrons, of negative ions, and of positive ions per unit volume.

Let the average time taken by an electron to acquire under the electric force sufficient energy to produce ionization by collision and to effect the ionization be T, then the number of ions produced by collisions per second in unit volume is n_1/T. The ionization due to radiation will be proportional to the radiation emitted in unit time; this will be proportional to the number of positive ions neutralized in that time, *i. e.* to $n_1 m$. We shall suppose that the rate of production of ions from radiation is $rn_1 m$. Hence the total rate of production of electrons per unit volume is

$$\frac{n_1}{\mathrm{T}} + rn_1 m.$$

This must be equal to the rate at which the electrons disappear; the electrons disappear by combining (1) with positive ions to form neutral molecules, (2) with neutral molecules to form negative ions. The rate of the first process is $a_1 n_1 m$, where a_1 is the coefficient of recombination between electrons and positive ions; the rate of the second is $a_2 n_1 \rho$, where a_2 is a coefficient of recombination and ρ the density of the gas in the discharge-tube. Hence equating the rate at which the electrons are produced to that at which they disappear,

$$\frac{n_1}{\mathrm{T}} + rn_1 m = a_1 n_1 m + a_2 n_1 \rho \quad . \quad . \quad . \quad . \quad (8)$$

The rate at which negative ions are produced is $a_2 n_1 \rho$, the rate at which they disappear is $\alpha n_2 m$, where α is the coefficient of recombination of the positive and negative ions and n_2 the number of negative ions per unit volume. Hence

$$a_2 n_1 \rho = \alpha n_2 m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In the uniform positive column the electric force is constant, so that there can be no resultant electric charge at any portion of it. Hence

$$n_1 + n_2 = m \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From (8), (9), (10) we get

$$m = \frac{\dfrac{1}{\mathrm{T}} - a_2 \rho}{a_1 - r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{n_1}{\alpha m} = \frac{n_2}{a_2 \rho} = \frac{m}{\alpha m + a_2 \rho}.$$

If i is the current through unit area, u_1, u_2, and v the velocities of the electrons, negative ions, and positive ions respectively, then

$$n_1u_1+n_2u_2+mv=\frac{i}{e},$$

or

$$\frac{\alpha u_1m^2+a_2u_2\rho m+\alpha vm^2+a_2v\rho m}{\alpha m+a_2\rho}=\frac{i}{e}. \quad . \quad . \quad . \quad (12)$$

This is an equation to determine m, and when m is known 1/T can be determined from (11). Now, as we shall see, T can be expressed in terms of the electric force X, so that (12) will lead to expression for the electric force.

When, as in the cases we are considering, the pressure is very low by far the greater part of the current is carried by the electrons. When this is so (5) reduces to

$$\frac{\alpha u_1m^2}{\alpha m+a_2\rho}=\frac{i}{e}.$$

For very small values of i this by equation (11) reduces to

$$\frac{1}{\mathrm{T}}=a_2\rho.$$

To find T we may proceed as follows. For the electron to ionize the gas by collision it must acquire under the electric field energy equal to that corresponding to the ionizing potential V. Let X be the electric force in the positive column, d the distance measured in the direction of the force through which the electron must move to acquire the energy corresponding to the ionizing potential, then, if there is no loss of energy at the collisions,

$$\mathrm{X}d=\mathrm{V}.$$

The electron while moving through a distance d from the source will make many collisions. The chance that after n collisions the distance it has travelled is between λx and $\lambda(x+dx)$ is $\frac{2}{n}\epsilon^{-\frac{x^2}{n}}\,x\,dx$. (Rayleigh, 'Theory of Sound,' 2nd ed. vol. i. p. 40). The average distance is equal to

$$\frac{2\lambda}{n}\int_0^\infty \epsilon^{-\frac{x^2}{n}}\,x^2\,dx$$
$$=\tfrac{1}{2}\pi^{1,2}\sqrt{n}\,.\,\lambda,$$

where λ is the mean free path of the electron. Hence n the

number of collisions it makes before it travels through a distance d may be taken as given by the equation

$$d = \tfrac{1}{2}\pi^{1/2}\sqrt{n}\,.\,\lambda.$$

Since the free path of an electron moving through uncharged molecules is proportional to its velocity (Phil. Mag. xlvii. p. 345) the time between each collision is constant. Hence T the time required to make these n collisions will be given by the equation

$$T = \frac{n\lambda}{v},$$

where v is the velocity and λ the free path of the unaccelerated ion. Now

$$n = \frac{4d^2}{\pi\lambda^2} = \frac{4}{\pi}\frac{V^2}{X^2\lambda^2},$$

hence

$$T = \frac{4}{\pi}\frac{V^2}{X^2\lambda^2}\frac{\lambda}{v}.$$

We now proceed to find $a_2\rho$; $1/a_2\rho$ is the average time taken by the electron to unite with a neutral molecule to form a negative ion. It follows from the theory of the recombination of ions (see Phil. Mag. xlvii. p. 360), and has been proved by the experiments of Loeb (Phil. Mag. xliii. p. 229) that an electron makes many collisions before uniting with a molecule to form a negative ion. Let N be this number, which has been measured by Loeb for several gases, then

$$\frac{1}{a_2\rho} = \frac{N\lambda}{v}.$$

Since, when the current is small

$$T = \frac{1}{a_2\rho},$$

$$\frac{4}{\pi}\frac{V^2}{X^2\lambda^2} = N,$$

or

$$X\lambda = \frac{2}{\sqrt{\pi}}\cdot\frac{V}{\sqrt{N}}.$$

The free path we are concerned with here is the path of a charged electron. In consequence of the charge λ will be shorter than the geometrical free path L, which is equal to $1/\rho\pi\sigma^2$, where σ is the radius of the uncharged molecule.

It can be shown (Phil. Mag. xlvii. p. 345) that

$$\frac{\lambda}{L} = \frac{1}{2\cdot 2}\sqrt{\frac{\beta\theta}{U}},$$

where $\beta\theta$ is the average energy due to thermal agitation of a molecule of the gas at the absolute temperature, θ, and U the work required to deprive a negatively electrified molecule of its electron. Hence

$$XL = \frac{4\cdot 4V}{\sqrt{\pi}}\sqrt{\frac{U}{N\beta\theta}}.$$

This equation is obtained on the supposition that i the current is very small.

When i is finite we have by (12)

$$\frac{\pi}{4}\frac{X^2\lambda^2}{V^2} = \frac{1}{N} + (a-r)\frac{\lambda}{v}\frac{i}{eu_1}.$$

The fact that an electron makes a very large number of collisions before it gets attached to an uncharged molecule reduces the electric force required to make an electron ionize the gas to a value much below that which would be required if the electron only made a few collisions when in the free state. For example, if the electric force is such that $X\lambda = V/10$, where X is the electric force and λ the mean free path, then, if the electron only makes one collision when in the free state, the chance of its getting enough energy to ionize is the chance of describing without collision a distance 10 times the mean free path, *i. e.*, ϵ^{-10}, a very small quantity. If it, on the other hand, makes 10^4 collisions in a free state, and if these collisions are elastic, the chance that after making these collisions the electron would have enough energy to ionize the gas is

$$\epsilon^{-\frac{10^2}{10^4}} \quad \text{or} \quad \epsilon^{-\frac{1}{100}},$$

which is very nearly a certainty.

Summary.

The paper contains a theory of the electric discharge at low pressures based on the view that the ionization in the dark space and in the negative glow is due (1) to the radiation emitted when the positive ions are neutralized at the cathode, this radiation by its photoelectric effect causes an emission of electrons from the cathode; these electrons fall through

the whole potential difference in the dark space and constitute the high-speed cathode rays. Experiments are described which prove that all the cathode rays have nearly the same energy, differing in this respect from the positive rays which have energies ranging over wide limits.

The direct ionization due to the collision of high-speed cathode rays is, as Glasson's experiments show, small at their low pressures, but though these rays in their passage through the gas do not liberate many electrons, they cause the molecules through which they pass to emit a radiation which ionizes the surrounding gas in the negative glow and the dark space, the luminosity of the glow being the result of the recombination of electrons and positive ions produced in this way. The recombination of these electrons and positive ions also gives out a radiation which can ionize gases whose ionizing potential is less than that of the neutral atom or molecule formed by the combination of the electron and positive ion.

The positive ions, except in light gases like hydrogen and helium, will not, unless the potential differences are of the order of sixty-five thousand volts for the oxygen atom and higher for other gases in proportion to their molecular weights, produce free electrons by collisions. This is due to the great differences between the masses of the ions and the electrons, which makes it impossible for more than a fraction of the energy of one to be transferred to the other by a collision.

An expression for the distribution of potential in the dark space is deduced from these principles; according to this expression the potential difference between a point in the dark space and the negative glow is proportional to the square of the distance of the point from the edge of the glow. This is the result arrived at by Dr. Aston in his experiments on this subject. An expression is also obtained for the connexion between the thickness of the dark space and the pressure of the gas.

Experiments are also described on the character of the ions in the positive column, and an expression found for the electric force in a uniform unstriated column.

II. *On the Helmholtz Theories of the Struck String.*—Part II. *Experimental.* *By* W. H. George, *B.Sc., Research Student, University College, Nottingham* *.

[Plates I.–III.]

IN a previous paper (Phil. Mag. xlvii. pp. 591–602 (1924)) the Helmholtz theories of the struck string were considered, and in order to subject the second theory to quantitative experimental test expressions were there deduced for quantities whose magnitude could be determined from a displacement-time curve of the struck point of the string. In the present paper the apparatus used for obtaining experimentally the required displacement-time curves is described together with the results obtained. As the apparatus has been used to investigate a number of problems of the struck string, it will be described here in such detail as to serve the purpose of future papers as well as the present one.

Description of Apparatus.

The String.—The steel piano wire used is stretched on a monochord for lengths up to 100 cm. and across bridges (B) fixed upon roof-posts of the laboratory for lengths up to 316 cm. Two advantages of the use of the long string are, first, the greater range of available values of the ratio of the mass of the hammer to the mass of the string (m/M), and, second, the larger displacements obtained for a given mass ratio.

The exact striking place can be found by smoking the string before taking a record and allowing the hammer to strike once only. The bright speck caused by the removal of the black deposit at the striking place gives a clearly defined zero mark from which the distance of the nearer bridge can be measured. When a wedge-shaped metallic hammer-head is used an alternative method available consists in smoking the hammer face only. After a single impact a tiny black speck is left on the bright string and serves as the zero mark.

The Hammer.—Previous workers † have used the mechanism of the pianoforte key. The hammer used in the present research may be regarded as a compound pendulum

* Communicated by Prof. E. H. Barton, F.R.S.

† Kaufmann, Wied. *Ann.* liv. (1895); Berry, Phil. Mag. xix. pp. 648–51 (1910), and xx. pp. 652–57, xxii. pp. 113–118 (1911).

to which simple calculations are applicable. Photographs of three types of these pendulums are shown in Pl. I. fig. 2. The shaft consists of a lath of wood at one end of which two boxwood side pieces are glued. These serve to support the steel axle which fits in a hole drilled accurately perpendicular to the plane of the lath. The method of suspension will be clear from the photograph. The hammer-head is usually fixed at two-thirds the length of the pendulum from the axle, in order to avoid violent reactions there. In some cases when it is desirable to reduce the mass of the pendulum, it is shortened and the hammer-head is fixed at the end. Wedge-shaped hammer-heads are made by folding and hammering aluminium or copper foil. They are each securely fastened to the shaft by two screws and nuts, so that relative motion between hammer-head and shaft is prevented. As these hammer-heads are not wide enough to cover the slit of light used in observing the motion of the struck point, pieces of foil are usually fixed on them so that, at the instant of contact, the foil is in a plane parallel to the plane of motion of the string and has one edge parallel to and vertically beneath the string. Any relative motion of the hammer-head and the string in or perpendicular to the direction of the string's length during the impact can be tested by previously smoking the string or the hammer-head. The felt-covered hammer-heads are machine made of the type used in pianofortes. The block of wood (A) from which the pendulums are suspended is secured to the cross-piece (C) by a single bolt and wing-nut so that the hammer strikes at any desired place along the string. The cross-piece (C) is adjustable in the two directions perpendicular to the string. It will be noted that the pendulum can be removed from the apparatus in order to study the damping with the suspension unaltered. The natural vibrations of the hammer-shaft can be studied by removing the pendulum from its suspension, withdrawing the steel axle, and clamping the boxwood side pieces in a heavy vice. Distortion of the hammer during the impact can be studied by instantaneous photography and also by evidence deduced from the displacement-time curve of the struck point.

Optical Arrangements.—Fig. 3 shows one portion of the optical arrangements. An arc and condenser serve to illuminate by a horizontal beam a vertical slit (S) formed by two safety-razor blades. This horizontal beam is changed to a vertical one by the prism (P) and a horizontal image of the slit is focussed by the lens (L) upon the string, the final image being perpendicular to the string. The whole of this

portion of the apparatus is carried by a platform which runs on rails (RR) parallel to the string. Referring now to fig. 1, the image of the slit, a portion of which is cut off by the string, is focussed by the lens (D) onto the photographic plate through a slit in the "camera." The magnification in the final image is 5/3.

The Camera.—This consists of a long shallow wooden box having a groove along each of its two longer sides. In these two grooves a plate-holder (H) slides and is shot by stretched catapult elastic at uniform speed past a slit in the lid of the box, the elastic being just unstretched as the right-hand edge of the plate reaches the slit. A spring device at the right-hand end of the box prevents the rebound of the plate-holder past the open slit. In fig. 1 (Pl. I.) the box is shown with the lid open. In an actual experiment the box is taken into a dark-room and a plate is put in the holder which is pulled to the left-hand end of the box and there held by a brass pin which passes up through the bottom of the box and through a hole in the plate-holder. The lid of the box is then closed onto a felt pad which renders the box light-tight, and the box is removed from the dark-room and placed against stops on a platform which is set so that the plate travels parallel to the string. Special Rapid or Press plates of speed H & D 270–475 were used.

Timing Arrangements.—In order to obtain a record of the impact it is necessary to have some timing device which will ensure the arrival of the plate at the slit just before the hammer strikes the string. To facilitate this, two electro-magnets (M and E) are used and are connected with a cell and a morse-key (K) so that when the key is depressed (i.) the circuit containing M is made, so releasing the hammer, and (ii.) the circuit containing E is broken, so releasing a weight which falls from an adjustable height (h) upon a lath (F). Any timing required is attained by the corresponding adjustment of this height (h) which can be approximately calculated, since the time of descent of the hammer can be obtained from its initial inclination with the vertical. The lath (F) is pivoted at one end and has at the other end a right-angled hook which fits into the brass pin of the camera previously mentioned. By this means the falling weight knocks out the pin and releases the plate.

The velocity (v_0) of the hammer at the beginning of the impact is adjustable by varying the initial inclination (β) of the hammer with the vertical. If the hammer-head is attached at two-thirds the length of the pendulum from the

axis of rotation, we have :

$$v_0 = \sqrt{2gr(1 - \cos\beta)},$$

where r is the distance between string and axis of rotation of the pendulum. If the hammer-head is attached at the end of the pendulum, then :

$$v_0 = \sqrt{3gr\frac{(m' + 2M')(1 - \cos\beta)}{(m' + 3M')}},$$

where m' is the ordinary mass of the hammer-shaft and M′ is the mass of the hammer-head. When the apparatus is set for taking a record, the pendulum is held at the desired inclination with the vertical by the pin which rests upon a small cylindrical metal rod which in turn slides in bearings and has a piece of iron attached to the other end opposite to the electromagnet (M). By this arrangement the pendulum rests under a minimum of constraint, so that when it is released by the electromagnet it descends to the string without vibrations having been set up in its shaft. During the impact we may regard the pendulum as vertical and its velocity as uninfluenced by the acceleration of gravity.

On the experimental displacement-time curves accompanying this paper (Pls. II. & III.) the time axis is horizontal and positive from right to left. On the original larger figures (*i. e.* Pl. II. figs. 1–9) 1 cm. along the time axis represents approximately 1/318 sec., whilst on the smaller figures (Pl. III. figs. 12–23) 1 cm. represents approximately 1/205 sec. The dark patch on each records the motion of the hammer which covers up the slit during the impact. From the inclination (α) with the time axis of the right-hand linear boundary of the patch the initial velocity (v_0) of the hammer can be determined, and is given by

$$v_0 = \mathrm{V}\tan\alpha/\text{magnification},$$

where V is the velocity of the plate. Similarly, the velocity of rebound (v) of the hammer can be determined.

The graph paper effect appearing on some of the figures (Pl. III. figs. 16–19) is obtained by the use of a grid-plate as described in 'Nature' (vol. cxiii. p. 387 (1924)).

Test of Apparatus.

Before applying the measurements to test the Helmholtz theory, it is desirable to compare them with similar measurements made by Kaufmann, whose work must be taken as

standard. In Table I. are given examples of the percentage differences between Kaufmann's experimental and theoretical values in a number of cases chosen by him to support his theoretical work.

TABLE I.

Page of Kaufmann's paper.	l/a.	m/M.	Quantity.	No. of Experiments.	% Difference.
693	2	0·917	$\frac{\tau}{\theta}$	1	− 4·58
694	2	,,		1	− 5·93
693	2	,,	y_2	1	−12·15
694	2	,,		1	−19·4
696	10	1·78		5	+ 1·6 to + 2·4
,, ...	11·25 to 4·5	1·385	$\frac{\tau}{\theta}$	4	−11·6 to + 2·8
,, ...	20·4	1·35 to 3·51		3	− 5·06 to + 4·17
*697	10·8	2·79		5	− 2·5 to +18·8

No measurements given for y_2 when $l/a > 2$.

* Throughout the paper an asterisk denotes the use of a felt-covered hammer.

From the above table it is clear that in work of this nature large differences between experimental and theoretical values must be expected. This is due firstly to the large number of quantities to be measured, and secondly to the fact that in order to treat the problem theoretically at reasonable length a number of factors have to be omitted.

In Table II. experimental values obtained in the present work are compared with the corresponding values obtained from Kaufmann's theory. By the term "% Difference" is here meant

$$\frac{\text{Expmtal. value} - \text{Theo. value}}{\text{Theo. value}} \times 100.$$

By comparison of the percentage difference columns in Tables I. and II. we may conclude that, although the apparatus differs from Kaufmann's, the results obtained with it are of the same order of accuracy as Kaufmann's results. It may be noted that in all cases it was found possible to obtain a second displacement-time curve identical with the first as tested by superposition of the negatives. Figs. 10 and 11, originally intended to illustrate two cases of this repetition of any desired curve, have been omitted for reasons of economy.

TABLE II.

Quantity.	Figure.	On Kaufmann's Theory.	By Experiment.	% Difference.
$\frac{\tau}{\theta}$	*1	0·567	0·57	nil
	2	0·567	0·57	nil
	3	0·989	1·0	1
	4	0·637	0·633	−0·6
	5	0·381	0·355	−6·8
	6	0·244	0·243	nil
$\frac{t_2}{\theta}$	*1	0·50	0·50	nil
	2	0·50	0·51	2
	3	0·564	0·563	nil
	4	0·318	0·305	− 4·0
	5	0·1907	0·17	−10·8
	6	0·124	0·115	− 7·4
y_2.	*1	0·895 cm.	1·0 cm.	11·7
	2	0·803	0·9	12·1
	3	3·55	4·1	15·5
	4	2·37	2·36	nil
	5	1·56	1·5	−4
	6	0·875	0·7	−20
$\frac{-v}{v_0}$	*1	0·66	0·696	5·5
	2	0·66	0·63	−4·5
	3	0·634	0·68	7·2
	4	0·691	0·69	nil
	5	0·7986	0·817	2·3
	6	0·450	0·4225	−6·1

TEST OF HELMHOLTZ THEORY.

In the following two tables experimental values are compared with those predicted by the Helmholtz theory and also where possible with those predicted by the Kaufmann theory.

TABLE III.

Quantity.	Fig.	By experiment.	On Helmholtz Theory.		On Kaufmann Theory.	
			Theo. Value.	% Difference.	Theo. Value.	% Difference
$\frac{t_2}{\theta}$	*1	0·50	0·318	57	0·500	nil
	2	0·51	0·326	56	0·500	2
	3	0·563	0·562	nil	0·564	nil
	4	0·305	0·320	− 4·7	0·318	− 4
	5	0·17	0·189	−10·0	0·1907	−10·8
	6	0·115	0·1375	−16·3	0·124	− 7·4
y_2.	*1	1·0 cm.	2·68 cm.	−63	0·895 cm.	11·7
	2	0·9	2·46	−63	0·803	12·1
	3	4·1	4·17	− 2	3·55	15·5
	4	2·36	2·25	4·9	2·37	nil
	5	1·5	1·647	− 8·9	1·56	− 4
	6	0·7	0·985	−29	0·875	−20

TABLE IV.

Quantity.	Fig.	By experiment.	On Helmholtz Theory.	% Difference.
t_2	7	0·312	0·2535	23
$\overline{\theta}$	8	0·359	0·262	37
	9	0·575	0·5085	13
	*12	0·52	0·608	−14
	13	0·774	0·825	− 6
	14	0·857	0·902	− 5
	15	0·96	1·075	−11
	*16	0·606	0·602	nil
	*17	0·625	0·593	5
	*18	0·6565	0·618	6
	*19	0·613	0·634	− 3
	*20	0·64	0·82	−22
	*21	0·64	0·82	−21
	*22	0·53	0·78	−32
	*23	0·58	0·83	−30
y_2	7	0·93 cm.	2·37 cm.	−61
	8	0·9	2·27	−60
	9	3·6	3·615	nil
	*12	1·1	1·39	−21
	13	1·7	1·91	−11
	14	1·7	2·175	−22
	15	1·8	1·705	6
	*16	0·50	0·6015	−17
	*17	0·95	1·083	−12
	*18	1·36	1·626	−16
	*19	1·65	2·22	−26
	*20	0·65	0·81	−20
	*21	1·1	1·56	−29
	*22	1·55	1·87	−17
	*23	1·95	2·88	−32

TABLE V.

			t_2/θ % Difference.		y_2 % Difference.	
m/M.	l/a.	Fig.	Helmholtz Theory.	Kaufmann Theory.	Helmholtz Theory.	Kaufmann Theory.
0·267	2	*1	57	nil	−63	11·7
,,	2	2	56	2	−63	12·1
,,	3	7	23	...	−61	...
,,	8/3	8	37	...	−60	...
,,	9	6	−16·3	−7·4	−29	−20
1·45	2	*20	−22	...	−20	...
,,	,,	*21	−21	...	−29	...
,,	,,	*22	−32	...	−17	...
,,	,,	*23	−30	...	−32	...
,,	20/7	*16	nil	...	−17	...
,,	,,	*17	5	...	−12	...
,,	,,	*18	6	...	−16	...
,,	,,	*19	−3	...	−26	...
1·684	2	3	nil	nil	−2	15·5
,,	8/3	9	13	...	nil	...
,,	9	4	− 4·7	− 4·0	4·9	nil
,,	26·3	5	−10·0	−10·8	−8·9	−4·0
7·47	9	15	−11	...	6	...
,,	10	14	− 5	...	−22	...
,,	11	13	− 6	...	−11	...
9·19	20	*12	−14	...	−21	...

TABLE VI. *Description of Figures.*

The length (l) of the string is 316 cm. for figs. 1–9 and is 100 cm. for figs. 12–23. The linear density of the string is 0·0174 gm. per cm. for figs. 12–15 and is 0·093 gm. per cm. for figs. 1–9 and 16–23. τ = duration of impact; θ = free period of fundamental tone of string; v_0 = initial velocity of hammer; v = velocity of rebound of hammer; a = distance of struck point from nearer bridge.

Fig.	l/a.	m/M.	m.	τ/θ.	v_0.	$-v$.
*1	2	0·269	7·9	0·57	377	262
2	„	„	„	„	338	213
3	„	1·684	49·5	1·0	334	227
4	9	„	„	0·633	361	249
5	26·3	„	„	0·355	375	306
6	9	0·267	7·9	0·243	352	149
7	3	„	„	0·392	391	150
8	8/3	„	„	0·422	358	166
9	„	1·684	49·5	1·0	308	291
*12	20	9·19	16	1·12	348	265
13	11	7·47	13	1·62	350	328
14	10	„	„	1·72	366	314
15	9	„	„	1·87	270	258
*16	20/7	1·45	13·5	1·15	122	106
*17	„	„	„	1·12	222	184
*18	„	„	„	1·14	320	246
*19	„	„	„	1·167	428	327
*20	2	1·45	13·5	1·64	120	120
*21	„	„	„	1·57	230	202
*22	„	„	„	1·45	292	229
*23	„	„	„	1·46	425	275

Summary of Experimental Results.—In Part I. of the whole paper it was shown that the first Helmholtz theory is in conflict with experimental evidence.

The experimental results obtained in the second part of the paper, which deals with the second Helmholtz theory, are summarized in Table V. where percentage differences between experiment and theory are classified according to the mass ratio of hammer and string (m/M) and striking place (l/a). Large differences between experiment and theory occur when a light hammer (m/M = 0·267) is used, but when the mass ratio of hammer and string is of the order 1·7 experiment and theory agree within the limits of experimental accuracy. In each of the two series figs. 16–19 and 20–23 the initial velocity of the hammer was varied whilst m/M and l/a were kept constant. In the complete set of figures the range of m/M is from 0·267 to 9·19 and of l/a from 2 to 26·3.

Plan of the Research.—The continuation of the work is on the following lines. (I.) Critical examination of the various theories of the struck string with quantitative experiments and comparisons. (II.) Experimental determination of the

pressure-time law of the impact graphically and by an electrical method. (III.) Determination of the velocity-time graph of the struck point by an electromagnetic method and its use in the study of the communication of energy to the system. (IV.) The study of the total energy communicated to the system by the hammer (by the method indicated in Phil. Mag. xlvii. p. 596 (1924)) and its partition among the various partials. (V.) The application of instantaneous photography to the study of the form of the whole string at any instant (*loc. cit.* p. 593).

Throughout the work the influence of at least the following six factors must be studied:—(i.) a/l ; (ii.) m/M ; (iii.) tension of the string; (iv.) v_0; (v.) dimensions and nature of the hammer-head ; and (vi.) vibration of the hammer-shaft.

In addition to previous acknowledgements the writer wishes to thank Dr. H. M. Browning, M.Sc., for occasional help in making experimental adjustments, and Mr. A. H. Franks, B.Sc., for kindly helping to photograph the apparatus.

University College, Nottingham.
February 1924.

Note added May 31st, 1924.

On account of the clearness of the reproduction of the displacement-time curves (figs. 1–23), it seems desirable to comment on some other of their features not mentioned in the paper.

Momentary separation of hammer and string.—In each of the figures 1 and 2 a small white patch can be seen at the top of the dark patch between the record of the string's motion and that of the hammer. This shows that during the impact there was complete separation and subsequent contact between hammer and string before the hammer finally quitted the string. This has an important bearing on the determination of the total duration of the impact by electrical methods. These methods, which are by far the most convenient and are widely used to study all cases of impacts between metallic or metal-coated bodies, depend in part on the assumption that the variation in electrical resistance of the contact due to variation in the mechanical forces involved are negligible as compared with the total electrical resistance of the circuit. It is clear that in this particular case such an assumption would be unjustifiable, since the electrical resistance has an infinite value for a portion of the total duration of the impact.

Fig. 1.

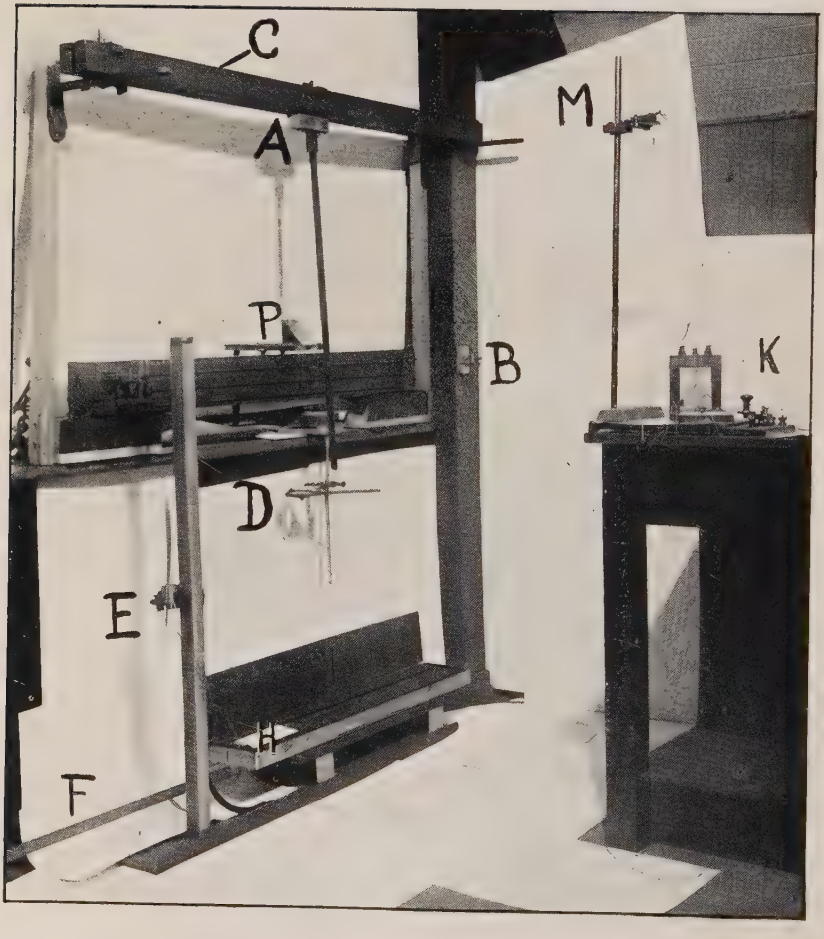


Fig. 2.

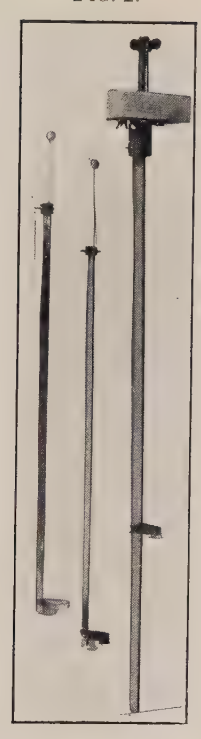
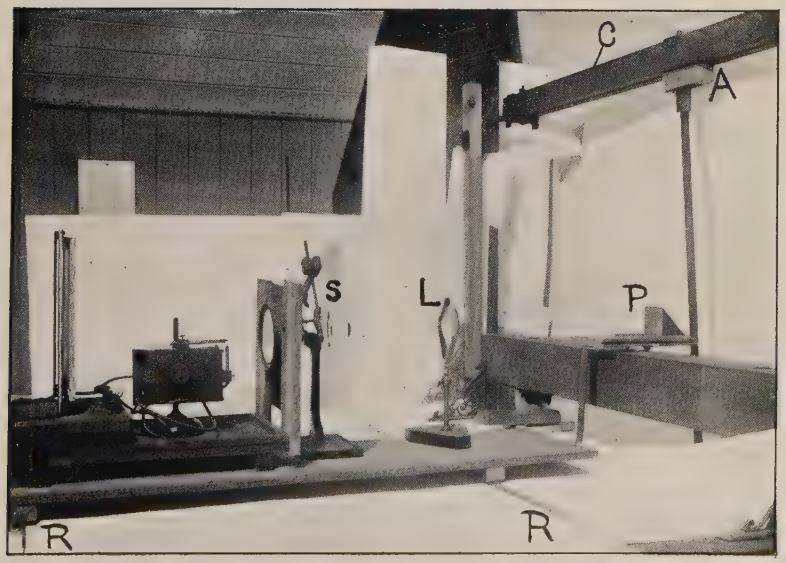
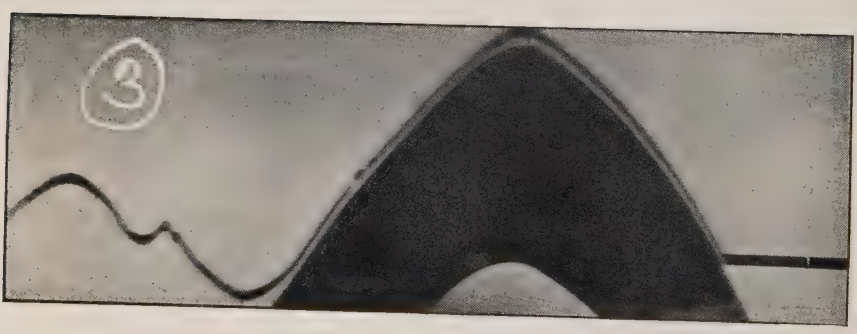


Fig. 3.

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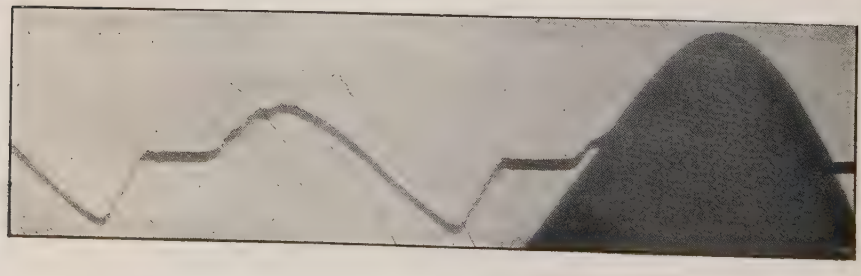

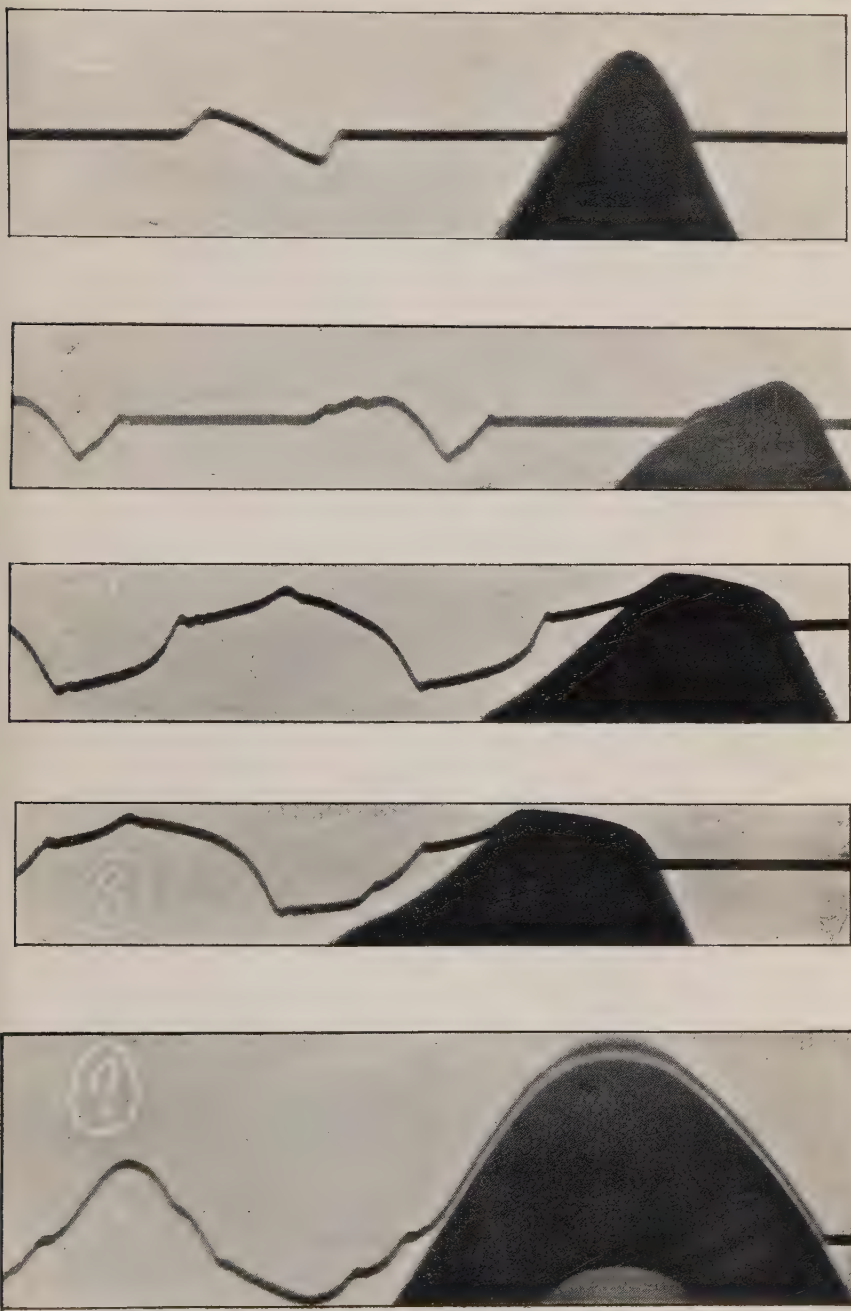

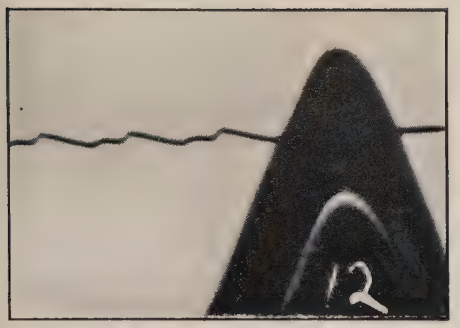
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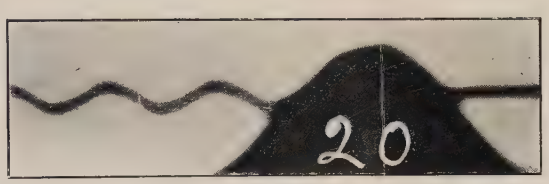
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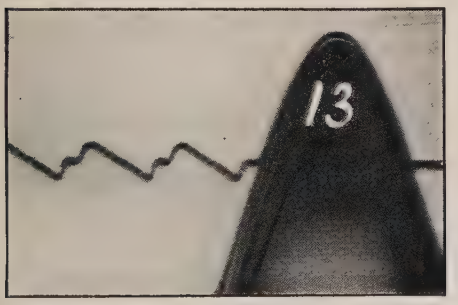
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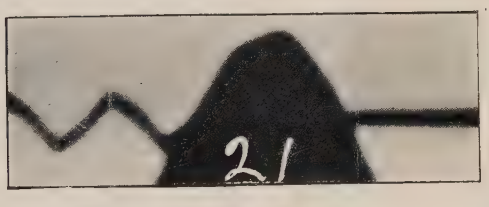
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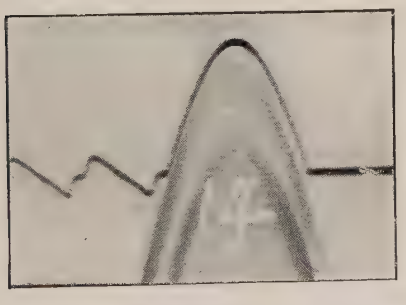

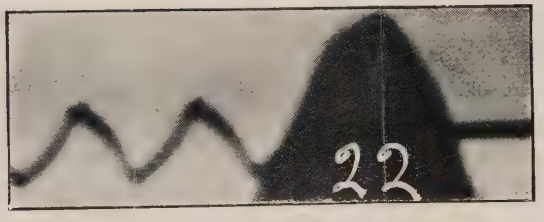
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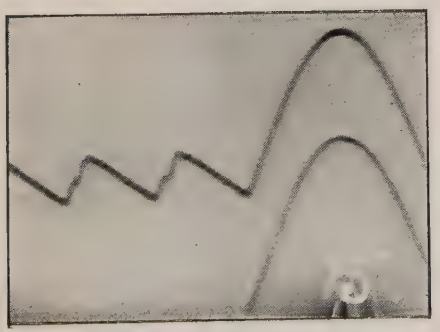

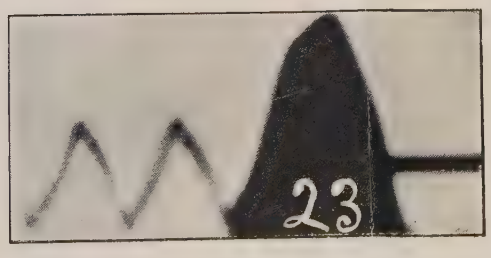
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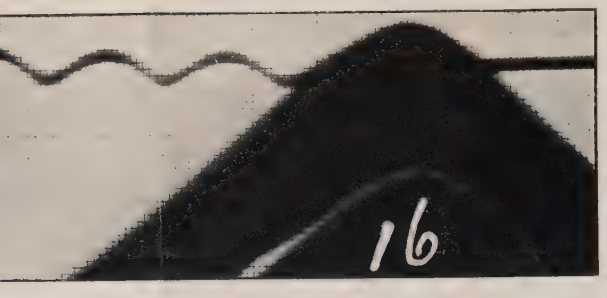
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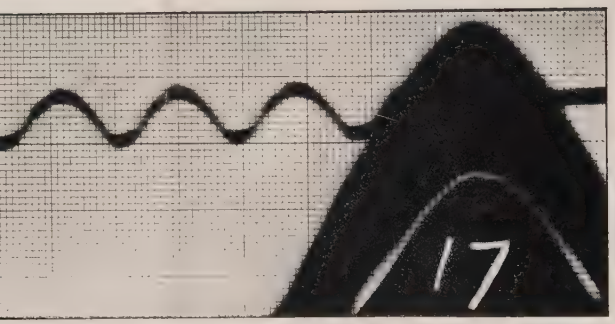
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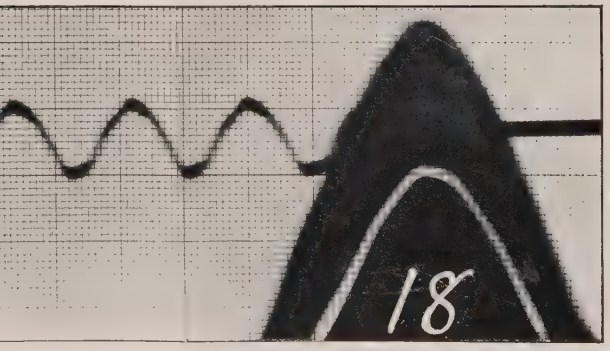
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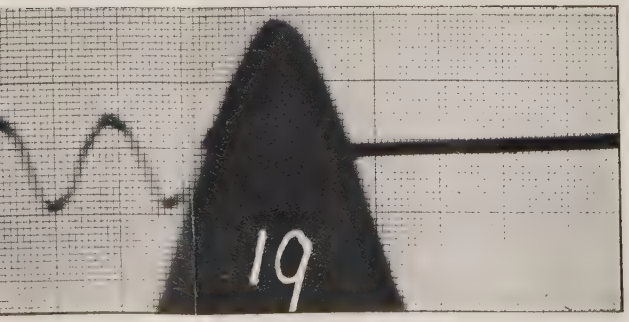
19

The present writer has investigated this separation phenomenon by the use of the Duddell Oscillograph. As might be expected, the effect depends on the mass ratio of hammer and string (m/M), and on the striking place (a/l). Experiments on the influence of other factors are now in progress.

On figs. 3 and 9 a white band about 1 mm. wide can be seen between the record of the string and that of the hammer. This is due to an inexact fixing upon the hammer-head of the strip of foil which cuts off the light. It does not, of course, indicate separation of the hammer and string.

Distortion of hammer.—The felt-covered hammer-heads used consist of wedge-shaped pieces of wood round which the felt is tightly wrapped by a machine process. In order to attempt a study of the compression of this felt a small hole was drilled in the narrow end of the wood behind the felt. If now the hammer approach and recede along the line of the slit of light a continuous white line will be seen on the dark patch caused by the hammer's shadow (figs. 17 and 18), and the extent of the compression of the felt can be estimated from the perpendicular distance between the white line and the upper boundary of the dark patch. Before interpreting the result, account must be taken of the produced or already existent curvature of the face of the felt. The break in the white line shown in fig. 1 indicates a slight torsional deflexion of the hammer, the light being cut off until the hammer-head reaches a second hole drilled further from the edge of the felt and a little to the right of the axis of the hammer-head.

Fig. 15 illustrates an alternative method of recording the motion of the metallic type of hammer. A piece of fine wire is stretched across the space between the metal foil of which the hammer is made, so that the wire is parallel to and in the same horizontal plane as the string. The lower line then records the motion of the hammer, whilst the upper line records the motion of the string. This arrangement needs a careful setting of the hammer relative to the image of the slit, otherwise the foil of the hammer may cause a blurred image as in fig. 14.

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III. *On Water Waves near the Shore.* *By* HAROLD JEFFREYS, *M.A., D.Sc., Fellow and Lecturer of St. John's College, Cambridge* *.

A GLANCE at a rough sea at a distance from land is enough to show that the waves form no regular pattern, but that as a rule the length of the crest of each wave is comparable with the distance between consecutive waves: thus waves in deep water constitute essentially a three-dimensional phenomenon. This fact appears to be consistent with the hypothesis that such waves are due to turbulence in the wind. For G. I. Taylor has shown that the mean square deviation of the velocity in any direction from its mean value is the same whether the direction be along the wind, across it, or vertical, and that gusts in any direction last about the same time; thus it would be expected that the irregularities set up in the motion of the sea would also have comparable wave-lengths in perpendicular horizontal directions.

Near the shore, however, this type of motion breaks down. Waves of the above type may be seen approaching the shore, but when they reach a distance from it, often of the order of fifty metres on a shelving beach, the crests are seen to run together transversely, and to form a series of waves with very long crests approximately parallel to the shore, which then approach the shore and break one by one.

It may be conjectured that this phenomenon is due either to a second-degree effect, due to the height of the waves becoming comparable with the depth of the water, or to the reduction of the depth of the water having a greater magnifying effect on the amplitudes of long-crested waves than on those of short-crested ones. It is shown here that the former alternative is correct.

Taking the former alternative first, let us consider the behaviour of progressive waves in water of uniform depth when the square of the amplitude is retained in the analysis and several types of wave are present. The equations of fluid motion are :—

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-g\frac{\partial \zeta}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-g\frac{\partial \zeta}{\partial y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial \zeta}{\partial t}=-\frac{\partial}{\partial x}\left\{(h+\zeta)u\right\}-\frac{\partial}{\partial y}\left\{(h+\zeta)v\right\}, \quad . \quad . \quad (3)$$

* Communicated by the Author.

where u, v are the horizontal components of velocity in the directions of x and y increasing, ζ is the vertical displacement of the surface, and h the depth of the water. If squares of u, v, and ζ are neglected it is easily seen that

$$\frac{\partial^2\zeta}{\partial t^2} = c^2\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $c^2 = gh$ (5). Hence any wave of form

$$\zeta = b \cos \lambda(x - dt) \cos \mu y \quad . \quad . \quad . \quad . \quad (6)$$

is an approximate solution, provided

$$\lambda^2 d^2 = c^2(\lambda^2 + \mu^2) . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The corresponding values of u and v are

$$u = \frac{gb}{d} \cos \lambda(x - dt) \cos \mu y, \quad . \quad . \quad . \quad . \quad (8)$$

$$v = -\frac{gb\mu}{\lambda d} \sin \lambda (x - dt) \sin \mu y. \quad . \quad . \quad . \quad (9)$$

If now two waves of type (6) are present in the motion, and we evaluate the terms of the second degree in (1) (2) and (3), these terms will be found to include some depending on each wave separately and also some product terms containing factors of the form

$$\begin{matrix}\cos \\ \sin\end{matrix}\left\{(\lambda_1 \pm \lambda_2)x - (\lambda_1 d_1 \pm \lambda_2 d_2)t\right\} \begin{matrix}\cos \\ \sin\end{matrix}\left\{\mu_1 \pm \mu_2\right\} y. \quad . \quad (10)$$

Such terms will satisfy (4) only if

$$(\lambda_1 d_1 \pm \lambda_2 d_2)^2 = c^2\{(\lambda_1 \pm \lambda_2)^2 + (\mu_1 \pm \mu_2)^2\}, \quad . \quad . \quad (11)$$

or, using (7), if

$$\lambda_1\lambda_2 d_1 d_2 = c^2(\lambda_1\lambda_2 \pm \mu_1\mu_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This relation is easily reduced to

$$\frac{\mu_1}{\lambda_1} \pm \frac{\mu_2}{\lambda_2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which case the two waves differ only in their horizontal scale ; the distances between consecutive crests along and across the direction of propagation are in proportion. For the present purpose such combinations are uninteresting. In general, then, the product terms in (1) (2) and (3) will be such as to give contributions to the solution that are purely

periodic functions of x, y, and t. They therefore do not imply any change of type in either wave as it advances.

In investigating possible changes in type of each wave it will therefore be enough to consider each wave separately by Airy's method. We have

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$$
$$=\frac{g^2b^2}{4\lambda d^2}\sin 2\lambda(x-dt)\{\mu^2(1-\cos 2\mu y)-\lambda^2(1+\cos 2\mu y)\}, \quad (14)$$

$$u\frac{\partial v}{\partial \lambda}+v\frac{dv}{dy}$$
$$=\frac{\mu g^2b^2}{4\lambda^2 d^2}\sin 2\mu y[\mu^2\{1-\cos 2\lambda(x-dt)\}-\lambda^2\{1+\cos 2\lambda(x-dt)\}], (15$$

$$\frac{\partial}{\partial x}(u\zeta)+\frac{\partial}{\partial y}(v\zeta)$$
$$=-\frac{gb^2}{2\lambda d}\sin 2\lambda(x-dt)\{\lambda^2(1+\cos 2\mu y)+\mu^2\cos 2\mu y\} . \quad (16)$$

Except in the case where μ is zero, the only terms in these expressions that can lead to non-periodic terms in the solution are those involving both $2\lambda(x-dt)$ and $2\mu y$. We have therefore to solve

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=\frac{\lambda g^2b^2}{4c^2}\sin 2\lambda(x-dt)\cos 2\mu y, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\frac{\partial v}{\partial t}+g\frac{\partial \zeta}{\partial y}=\frac{\mu g^2b^2}{4c^2}\cos 2\lambda(x-dt)\sin 2\mu y, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\frac{\partial \zeta}{\partial t}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{dy}\right)=\frac{\lambda dgb^2}{2c^2}\sin 2\lambda(x-dt)\cos 2\mu y. \quad . \quad . \quad (19)$$

Eliminating u and v we find

$$\frac{\partial^2\zeta}{\partial t^2}-c^2\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)=-\frac{3}{2}\frac{\lambda^2d^2gb^2}{c^2}\cos 2\lambda(x-dt)\cos 2\mu y, (20)$$

the relevant particular integral of which is

$$\zeta=\frac{3}{8}\frac{\lambda d^2gb^2}{c^4}x\sin 2\lambda(x-dt)\cos 2\mu y. \quad . \quad . \quad . \quad (21)$$

Thus the second approximation to the wave form is

$$\zeta=b\cos\lambda(x-dt)\cos\mu y+\frac{3}{8}\frac{\lambda d^2gb^2}{c^4}x\sin 2\lambda(x-dt)\cos 2\mu y. \quad (22)$$

The effect of the second term is to make the crests gain on the troughs as the wave advances. The approximation will evidently break down when $\frac{3}{8}\frac{\lambda d^2 gb}{c^4}x$ becomes comparable with unity. When this is the case the front of the wave has become so steep that the vertical acceleration cannot be neglected in the equation of vertical motion ; the crest then curls forward and pours over into the trough, and the wave is destroyed.

If in addition to the above wave a wave with a long crest is present, it will be necessary to include a term $a\cos\kappa(x-ct)$ in (6), and then a term $\frac{3}{4}\frac{g\kappa a^2}{c^2}x\sin 2\kappa\,(x-ct)$ will also have to be included in (22) *. The factor $\frac{3}{4}$ arises instead of $\frac{3}{8}$ on account of the fact that μ is here zero, so that the terms independent of y in (14), (15), and (16) have to be included in the solution. The distance this wave can travel before it collapses is therefore to the distance the former wave can travel in the ratio $\frac{\lambda d^2 b}{2\kappa a c^2}$. Now d^2 is greater than c^2, but comparable with it so long as μ is not large compared with λ, and therefore as a rule this ratio is greater than $\lambda b/2\kappa a$.

Now we have seen that far from shore the long-crested wave is so small as to be almost unobservable, whereas the short-crested waves are striking. Hence b is much greater than a. Also the short-crested waves appear to have the shorter wave-length, so that $\lambda>\kappa$. On both grounds, therefore, the critical ratio must be large. Hence as a compound wave advances in shallow water the short-crested constituents tend to annihilate themselves by breaking, while the long-crested wave does not break until all the shorter waves have disappeared. The phenomenon of the predominance of the long-crested wave close to shore is therefore explicable.

If the depth be variable, but the amplitude small compared with the depth, equations (1) (2) and (3) give, when u and v are eliminated, the equation

$$\frac{\partial^2\zeta}{\partial t^2}=g\left\{\frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)\right\}. \quad . \quad . \quad . \quad (23)$$

Suppose h to be a function of x alone, and search for a

* Lamb, 'Hydrodynamics,' 1906, p. 264.

solution of (23) of the form $X \cos \gamma t \cos \mu y$, where X is a function of x alone. Then

$$\frac{d}{dx}\left(h\frac{dX}{dx}\right)+(\gamma^2-gh\mu^2)X=0. \quad . \quad . \quad . \quad (24)$$

By hypothesis the wave is of harmonic type and is generated in deep water. Hence in deep water γ^2 is not less than $gh\mu^2$. As the wave passes into shallower water γ^2 remains the same but h diminishes; therefore $\gamma^2-gh\mu^2$ is positive. Thus X is an oscillating function of x. If we put

$$\int\frac{dx}{h}=\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

(24) becomes

$$\frac{d^2X}{d\theta^2}+(\gamma^2-gh\mu^2)hX=0, \quad . \quad . \quad . \quad . \quad (26)$$

and provided that h does not change by a large fraction of itself within a wave-length the amplitude of X will be proportional to $\{(\gamma^2-gh\mu^2)h\}^{-1/4}$. Thus, subject to the condition that the wave has come from deep water, the amplitude in shallow water will behave nearly like $h^{-1.4}$ for all waves. Therefore the second alternative offered above, that the long-crested waves are specially magnified in shallow water, does not afford an explanation of the phenomenon.

IV. *On the Delemer and the Lamb Theories of the Struck String.* *By* W. H. George, *B.Sc., University College, Nottingham* *.

IN the study of the struck string the pressure-time law of the impact is of considerable interest, since, if it be completely known, then the intensities of the fundamental tone and partials of the freely vibrating string, and therefore the resulting tone quality, can readily be determined by the well-known analytical method †. Three forms of the pressure-time law have been assumed by Helmholtz, by Delemer, and by Lamb respectively ‡. The present paper is concerned with experimental tests of the two latter theories, the Helmholtz theories having been already considered in two previous

* Communicated by Prof. E. H. Barton, F.R.S.

† Rayleigh, 'Theory of Sound,' vol. i. §§ 128 & 130.

‡ Helmholtz, 'Sensations of Tone,' App. V.; Delemer, Sci. Abs. A, vol. ix. p. 623 (1906); Lamb, 'Dynamical Theory of Sound,' p. 74 (1910).

papers*. The treatment is similar to that adopted in the first two papers. Expressions are derived for quantities whose magnitude can be determined from photographically obtained displacement-time curves for the struck point of the string, and the theoretical results are then compared with those found by experiment.

Theory.

Notation :—

l = length of string (in cm.).
a = distance of struck point from nearer bridge (in cm.).
y = variable displacement of struck point reckoned positive in the direction of the approaching hammer (in cm.).
y_1 = displacement of struck point at the instant of separation of hammer and string (in cm.).
y_2 = maximum displacement of struck point (in cm.).
r = distance between string and axis of rotation of hammer (in cm.).
θ = free period of fundamental tone of string (in sec.).
τ = duration of impact (in sec.).
t = variable time reckoned from the instant of the beginning of the impact (in sec.).
t_2 = time taken for y to reach its maximum value (in sec.).
v_0 = initial velocity of hammer (in cm. per sec.).
v = velocity of hammer at instant of separation of hammer and string (in cm. per sec.).
μ = linear density of string (in gm. per cm.).
m = mass of hammer (see following note).
M = mass of string $= \mu l$ (in gm.).

The actual hammer under consideration may be regarded as a compound pendulum. By the term "*mass of hammer*" is meant the mass in gm. of a particle equimomental with the hammer and distant r from the axis of rotation of the pendulum. The motion is assumed to take place in one plane passing through the string.

Delemer Theory.—The pressure of the hammer is assumed to be *constant* during the time of contact. We may therefore write for the force on the hammer

$$m\frac{d^2y}{dt^2} = \mathrm{K}. \qquad (0<t<\tau) \qquad \ldots\ldots\ldots \quad (1)$$

* Phil. Mag. xlvii. pp. 591–602 (1924), and *suprà*, p. 34. These will be referred to as the first and second papers respectively.

Hence

$$\int_{v_0}^{\frac{dy}{dt}} d\left(\frac{dy}{dt}\right) = \frac{K}{m}\int_0^t dt,$$

giving

$$\frac{dy}{dt} = v_0 + \frac{K}{m}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the quantity K is not readily determined by experiment, we may eliminate it from the equations by introducing the new quantity (v), which is the velocity of rebound of the hammer (see first paper, p. 596). By putting $t=\tau$ in equation (2) we have :—

$$v = v_0 + (K\tau)/m \quad \text{or} \quad K = m(v-v_0)/\tau.$$

Hence equation (2) may be rewritten thus :—

$$\underset{(0<t<\tau)}{\frac{dy}{dt}} = v_0 + (v-v_0)\frac{t}{\tau}. \quad . \quad . \quad . \quad . \quad (2\,a)$$

Integrating again and putting in the condition that $y=0$ when $t=0$, we have

$$\underset{(0<t<\tau)}{y} = v_0 t - \tfrac{1}{2}(v_0-v)\frac{t^2}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By putting $t=\tau$ in this equation, we see that the displacement of the struck point at the instant of separation of hammer and string is given by

$$y_1 = \tfrac{1}{2}(v_0+v)\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is identical with the expression for y_1 derived from the second Helmholtz theory (eqn. (5) of the first paper).

The time (t_2) taken for y to reach its maximum value (y_2) is obtained from equation (2 *a*), whence

$$t_2 = \frac{\tau v_0}{v_0-v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$y_2 = \frac{1}{2}\frac{\tau v_0^2}{v_0-v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Lamb Theory.—In this case we may write for the force on the hammer :—

$$m\frac{d^2y}{dt^2} = -\frac{2}{\pi}\cdot\frac{I\tau}{t^2+\tau^2}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where I represents the time-integral of the force from

$t=-\infty$ to $t=+\infty$. Proceeding as before, we have:—

$$\int_{v_0}^{\frac{dy}{dt}} d\left(\frac{dy}{dt}\right) = \frac{-2\mathrm{I}}{\pi m}\int_0^t \frac{\tau\, dt}{(t^2+\tau^2)}$$

or

$$\frac{dy}{dt} = v_0 - \frac{2\mathrm{I}}{\pi m}\tan^{-1}\frac{t}{\tau}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since t/τ is never greater than unity. To eliminate the quantity I, we put $t=\tau$ in equation (8), whence

$$v = v_0 - \mathrm{I}/(2m) \quad \text{or} \quad \mathrm{I} = 2m(v_0-v).$$

Hence equation (8) may be re-written:—

$$\underset{(0<t<\tau)}{\frac{dy}{dt}} = v_0 - \frac{4}{\pi}(v_0-v)\left(\tan^{-1}\frac{t}{\tau}\right). \quad . \quad . \quad . \quad (8\,a)$$

Integrating again with respect to time and putting in the condition that $y=0$ when $t=0$, we have

$$y = v_0 t - \frac{4}{\pi}(v_0-v)\left[t\tan^{-1}\frac{t}{\tau} - \frac{\tau}{2}\log_\epsilon\left(1+\frac{t^2}{\tau^2}\right)\right]. \quad (9)$$

Putting $t=\tau$ in this equation, we have

$$y_1 = \tau\left\{v_0 - (v_0-v)\left[1-\frac{2}{\pi}\log_\epsilon 2\right]\right\} \quad . \quad . \quad (10)$$

Putting $dy/dt=0$ in equation (8 *a*), we find

$$t_2 = \tau\tan\left[\frac{\pi v_0}{4(v_0-v)}\right]. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence the maximum displacement (y_2) of the struck point is given by:—

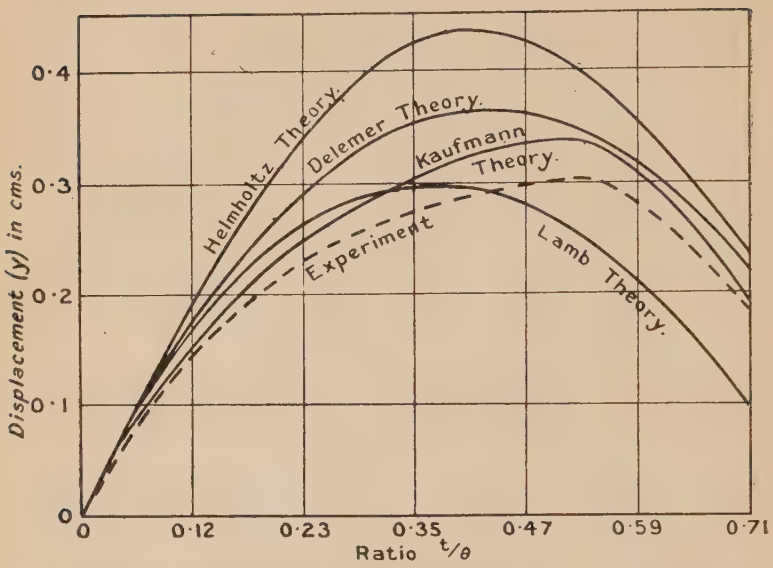
$$y_2 = \frac{-4\tau(v_0-v)}{\pi}\log_\epsilon\cos\left\{\frac{\pi v_0}{4(v_0-v)}\right\} \quad . \quad . \quad (12)$$

The corresponding expressions derived from the Helmholtz and the Kaufmann theories were given in the first paper.

Experimental Results.

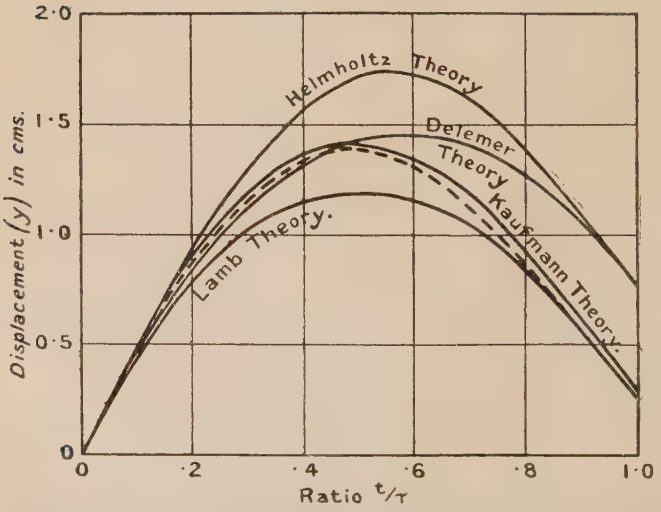
The measurements required in order to compare experiment and theory were made upon the displacement-time curves reproduced in Pls. II. & III. of the second paper.

Fig. 1.



Comparison of theoretical displacement-time curves for the struck point with the experimental curve. M=4·8; m=4·4; θ=1/128; τ/θ=0·74; v_0=215; v=−149; l/a=2.

Fig. 2.



Comparison of theoretical displacement-time curves for the struck point with the experimental curve. M=29·4; m=49·5; m_0=50·6; θ=1/46·1; τ/θ=0·633; v_0=361; v=−249; l/a=9.

TABLE I.

Fig.	t_2/θ by experiment.	t_2/θ on Delemer Theory.	t_2/θ on Lamb Theory.	% Difference on Delemer Theory.	% Difference on Lamb Theory.
1*	0·50	0·336	0·285	49	75
2	0·51	0·350	0·298	46	71
3	0·56	0·596	0 504	− 5	12
4	0·31	0·375	0·317	−19	− 4
5	0·17	0·195	0·163	−13	4
6	0·12	0·170	0·150	−32	−23
7	0·31	0·283	0·249	10	25
8	0·36	0·289	0 279	24	28
9	0·57	0·514	0·427	12	35
12*	0·52	0·636	0·537	−18	− 3
13	0·77	0·836	0·696	− 7	11
14	0·86	0·926	0·774	− 7	11
15	0·96	0·955	0·795	nil	21
16*	0·61	0·615	0·516	− 1	17
17*	0·62	0·613	0·513	2	22
18*	0·66	0·643	0·542	2	21
19*	0·61	0·662	0·558	− 7	10
20*	0·64	0·820	0·678	−22	− 6
21*	0·64	0·836	0·697	−23	− 8
22*	0·53	0·813	0·682	−35	−22
23*	0·58	0·887	0·752	−35	−23

TABLE II.

Fig.	y_2 by experiment.	y_2 on Delemer Theory.	y_2 on Lamb Theory.	% Difference on Delemer Theory.	% Difference on Lamb Theory.
1*	1·0	2·26	1·84	−56	−46
2	0·9	2·10	1·72	−57	−48
3	4·1	3·53	3·46	16	18
4	2·36	3·15	1·99	−25	19
5	1·5	1·34	1·30	12	16
6	0·7	1·12	0·93	−38	−25
7	0·93	2·03	1·69	−54	−45
8	0·9	1·89	1·88	−52	−52
9	3·6	2·87	2·32	25	55
12*	1·1	1·15	0·94	− 4	17
13	1·7	1·52	1·23	12	38
14	1·7	1·76	1·35	− 3	26
15	1·8	1·34	1·08	34	66
16*	0·5	0·49	0·40	2	26
17*	0·95	0·88	0·72	8	32
18*	1·36	1·34	1·09	1	25
19*	1·65	1·84	1·50	−10	10
20*	0·65	0·64	0·52	2	26
21*	1·1	1·25	1·02	−12	8
22*	1·55	1·55	1·25	nil	24
23*	1·95	2·45	2·00	− 2	2

Table III.

$\frac{m}{M}$	$\frac{l}{a}$	Fig.	t_2/θ % Differences.				y_2 % Differences.			
			Helmholtz Theory.	Kaufmann Theory.	Delemer Theory.	Lamb Theory.	Helmholtz Theory.	Kaufmann Theory.	Delemer Theory.	Lamb Theory.
0·267	2	1*	57	Nil	49	75	−63	12	−56	−46
	2	2	56	2	46	71	−63	12	−57	−48
	2	7	23	...	10	25	−61	...	−54	−45
	8/3	8	37	...	24	28	−60	...	−52	−52
	9	6	−16	− 7	−32	−23	−29	−20	−38	−25
1·45	2	20*	−22	...	−22	− 6	−20	...	2	26
	,,	21*	−21	...	−23	− 8	−29	...	−12	8
	,,	22*	−32	...	−35	−22	−17	...	Nil	24
	,,	23*	−30	...	−35	−23	−32	...	− 2	2
	20/7	16*	Nil	...	− 1	17	−17	...	2	26
	,,	17*	5	...	2	22	−12	...	8	32
	,,	18*	6	...	2	21	−16	...	1	25
	,,	19*	− 3	...	− 7	10	−26	...	−10	10
1·684	2	3	Nil	Nil	− 5	12	− 2	16	16	18
	8/3	9	13	...	12	35	Nil	...	25	55
	9	4	− 5	− 4	−19	− 4	5	Nil	−25	19
	26·3	5	−10	−11	−13	4	− 9	− 4	12	16
7·47	9	15	−11	...	Nil	21	6	...	34	66
	10	14	− 5	...	− 7	11	−22	...	− 3	26
	11	13	− 6	...	− 7	11	−11	...	12	38
9·19	20	12*	−14	...	−18	− 3	−21	...	− 4	17
Mean % Difference			1	...	− 4	13	−24	...	−10	9
Mean % Difference for figs. 1–6			14	− 3	4	22	−26	3	−25	−11

There also will be found details of the apparatus used and the necessary data for the calculations. In figs. 1 and 2 of the present paper typical experimental displacement-time curves of the struck point are compared with the curves predicted by four different theories. The experimental curve in fig. 1, which refers to a string struck at its mid-point, is taken from Kaufmann's paper (p. 693), whilst that in fig. 2, which refers to a string struck near one end, is taken from the curve reproduced in Pl. II. fig. 4 of the second paper. In each figure the experimental curve is shown dotted.

In each of Tables I. and II. quantities determined by experiment are compared with those predicted by the Delemer and the Lamb theories. The values of y_2 given in Table II. are the magnified values of the actual displacement of the struck point of the string, but the displacements shown in the graphs of fig. 2 represent the actual displacements of the struck point. This explains the apparent discrepancy between the value of y_2 given in the table and that shown on the graph. By the term "*percentage difference*" is meant throughout the paper [100(Expmtal. Value—Theo. Value)] ÷ (Theo. Value). An asterisk denotes the use of a felt-covered hammer-head. The absence of an asterisk denotes the use of a wedge-shaped metallic hammer-head.

The experimental results obtained in this and the second paper are summarized in Table III., where percentage differences between experiment and theory are classified according to the mass ratio of hammer and string (m/M) and striking place (l/a), these being the two most important factors. In each of the two series, 16–19 and 20–23, the initial velocity (v_0) of the hammer was varied, whilst m/M and l/a were kept constant. In the complete set of figures the range of m/M is 0·267–9·19 and of l/a is 2–26·3.

Acknowledgments and the plan of the continuation of the research were given at the end of the second paper.

University College,
Nottingham,
March 1924.

V. *Ionization, Excitation, and Dissociation of Gases.* *By* A. Ll. Hughes, *Wayman Crow Professor of Physics, Washington University, Saint Louis* †.

THE purpose of this communication is to call attention to a number of researches recently published on effects involving ionization, excitation, and dissociation in gases, to point out apparent difficulties in correlation, and to draw certain tentative conclusions. The investigations to be referred to deal almost entirely with hydrogen. The ionizing and radiating potentials of hydrogen have been studied by numerous physicists. Among the published results, perhaps those of Franck, Knipping and Kruger ‡ stand out as being most directly accounted for by theory. According to these investigators, hydrogen has the following critical potentials:—

I. P.	R. P.	Origin.
11·5	—	Molecular ionization (H_2^+).
—	13·6	Dissociation and excitation (H and H*). (3·5+10·2=13·7 volts.)
17·1	—	Dissociation and ionization (H and H^+). (3·5+13·5=17·0 volts.)
30·4	—	Dissociation and ionization of both atoms (H^+ and H^+). (3·5+13·5+13·5=30·5 volts.)

The symbol H* may conveniently denote an excited atom.

Similar results have been obtained by Olmstead § and by Horton and Davies ‖, except that the corresponding voltages are about a volt lower. Horton and Davies, however, do not find ionization near 11·5 volts. In addition to the critical potentials listed by Franck, Kruger and Knipping, they find molecular ionization (H_2^+) at 22·8 volts, and dissociation plus atomic ionization plus atomic excitation (H^+ and H^*) at 26·1 volts. (One might have expected them to find a critical potential at 22·8 volts corresponding to H* and H*. However, they find the critical potential at 22·8 volts to be an ionizing potential for the molecule.)

† Communicated by the Author.

‡ Franck, Knipping & Kruger, *Verh. d. Deutsch Phys. Ges.* xxi. p. 728 (1919).

§ Olmstead, Phys. Rev. xix. p. 189 (1922).

‖ Horton & Davies, Phil. Mag. xlvi. p. 872 (1923).

Thus, those critical potentials involving dissociation are explained quantitatively in terms of Bohr's theoretical values for the ionizing and radiating potentials of the atom, together with a value for the energy required to dissociate the hydrogen molecule. The value which suits Franck, Knipping and Kruger's results is 3·5 volts, while Olmstead's results and those of Horton and Davies require a value about 2·4 volts.

In all these investigations, the interpretation of any particular critical potential as a molecular effect, or as an atomic effect accompanied by dissociation, is not the result of direct experimental demonstration, but is inferred from the suggestions of theory and the hints given by the spectroscopic evidence. The application of positive-ray analysis by Smyth * to ionizing potential experiments introduced a direct experimental demonstration of the nature of the positive ions produced at different ionizing potentials. Smyth's results on hydrogen do not bear out the conjectures of the foregoing authors as to the nature of the ionizing potential at about 16 to 17 volts. Smyth found that H_2^+ appeared alone at this potential without a trace of H^+. Atomic ions (H^+) first appeared at 20·7 volts. Thus the previous interpretation of the process started at the 16–17 volts ionizing potential is incorrect. Smyth's work also shows that even with energies of impact running as high as 600 volts, the proportion of H^+ ions to H_2^+ ions is exceedingly small. No definite statement is made, but fig. 2 in his paper † suggests that H^+/H_2^+, *i.e.* the ratio of the number of atomic ions to the number of molecular ions, is considerably less than 1 in 20 ; perhaps it is as small as 1 in 50. Fig. 3 in the same paper suggests that at 22 volts H^+/H_2^+ is less than 1 in 200. For nitrogen the ionizing potential at 16·9 volts is an ionization of the molecule, while atomic ions do not appear until 27·0 volts. Also the ratio N^+/N_2^+ is less than 15 per cent. up to 370 volts impact energy. Smyth found that in hydrogen the number of atomic ions relative to the number of molecular ions increased very much with the pressure, showing that atomic ions could be obtained plentifully as a secondary process (*e. g.* a collision between a positive molecular ion and a neutral molecule). However, when the pressure was so low that the secondary processes were negligible, then the yield of atomic positive ions as compared with the yield of molecular ions was exceedingly small, so that in

* Smyth, Proc. Roy. Soc. cv. p. 116 (1923); civ. p. 121 (1923).

† Smyth, Proc. Roy. Soc. cv. p. 116 (1923).

the great majority of cases the direct result of a collision between an electron and a molecule is the formation of molecular ion, and not the dissociation of a molecule accompanied by the formation of an atomic ion. Similar results were obtained by Dempster* also by a positive-ray analysis method.

The writer† carried out an investigation two years ago on the dissociation of hydrogen and nitrogen by electron impacts as a function of the energy of impact. The experimental arrangement was a very simple one. An oxide-coated filament was mounted axially in a glass tube lined with nickel gauze, so that electrons could be accelerated from the filament to the gauze by an applied potential. Hydrogen or nitrogen was introduced at a pressure sufficiently low to ensure that an electron would rarely make two collisions in its journey from the filament to the gauze. Provided that the accelerating voltage was above 13 volts for hydrogen and above 17 volts for nitrogen, there was a progressive decrease in pressure as the electron current passed across the tube. Following Langmuir's extensive work on dissociation of hydrogen by heat, this decrease in pressure was interpreted as due to the condensation of atoms produced by the dissociating collisions on the walls of the tube, which was immersed in liquid air. From the rate of disappearance of the gas, the electron current, the dimensions of the tube, etc., the number of collisions resulting in dissociation could be calculated for each accelerating voltage. The curves for hydrogen and nitrogen are shown in figs. 1 and 2. (The ordinates in these curves have been recomputed on the basis of recent values for the collision frequency of electrons in hydrogen and nitrogen, viz. 12·24 and 23·53, in place of the older values assumed in the paper referred to.) It will be seen that the number of collisions resulting in dissociation in hydrogen is as much as 1 in 3 at the most favourable energy of impact (about 100 volts). In view of Smyth's and Dempster's observations that atomic ions can be obtained plentifully as a secondary process, it is necessary to make sure that the dissociations measured in this research are due directly to the primary electron impact, and not to any secondary process. If the dissociations observed were the direct and immediate result of a collision between a molecule and an electron, then the yield of dissociating collisions in terms

* Dempster, Phil. Mag. xxxi. p. 434 (1916); Phys. Rev. viii. p. 651 (1916).

† Hughes, Phil. Mag. xli. p. 778 (1921).

of the total number of collisions should be a constant independent of the pressure. If, on the other hand, the dissociation were the result of a secondary action, such as a collision between a normal molecule and an ionized molecule, then the yield of dissociating collisions in terms

Fig. 1.

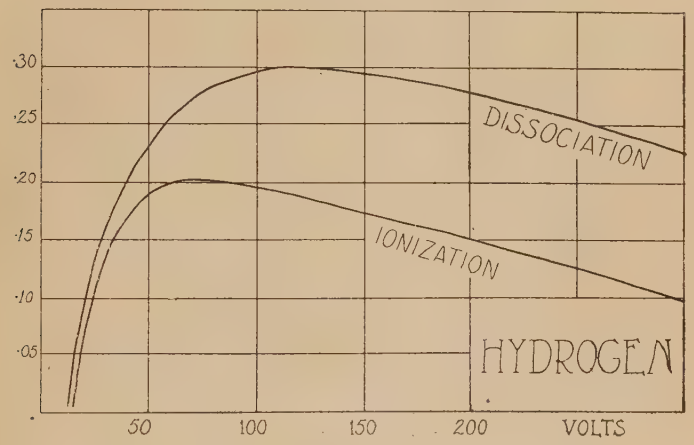
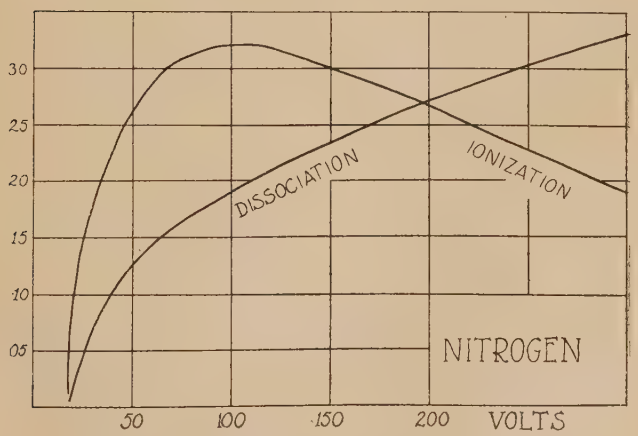


Fig. 2.



of the total number of collisions between electrons and molecules should increase directly as the pressure. In the experiments referred to, it was found that the fraction of collisions resulting in dissociation was practically constant and independent of the pressure. (In fact, the tendency was such as to suggest that the *lower* the pressure, the

greater the yield of dissociating encounters. This, however, could be explained by the fact that the glass takes up atoms less and less readily as it gets more and more covered by atoms.) It seems clear, therefore, that what was measured in these experiments is the fraction of collisions resulting in dissociation as a direct effect of an encounter between an electron and a molecule. Again, the dissociation effect starts at 13 volts for hydrogen and 17 volts for nitrogen. As both these are recognizable as critical potentials for these gases, and as no dissociation of hydrogen occurs at 3·5 volts (the value given by Langmuir for the amount of energy necessary to dissociate a hydrogen molecule), it is natural to conclude that dissociation by electron impact cannot occur alone. Dissociation by electron impact is to be regarded as always accompanied by simultaneous ionization, or by simultaneous excitation of one (or both) atoms.

The diagrams also contain curves (named "ionization") showing the fraction of collisions resulting in the formation of ions (atomic and molecular both count in) at different energies of impact of electrons. (This work was done by the author and Dr. Klein †.) There is a great similarity between the dissociation and ionization curves in the case of hydrogen, but none in the case of nitrogen.

We are now in a position to compare the writer's results on dissociation with those of Smyth on the nature of the positive ions. In the case of hydrogen, at the most favourable energy of impact, fig. 1 shows that about one collision in every three results in dissociation. If this dissociation is accompanied by ionization, then the yield of atomic ions cannot be less than half of the yield of molecular ions, which presupposes that the two remaining non-dissociating collisions both result in molecular ions. However, Smyth's experiments show that the number of atomic ions produced is only a very small fraction of the number of molecular ions. Hence it seems necessary to conclude that the dissociation in my experiments was not accompanied by atomic ionization. The possible direct effects of a collision between an electron and a molecule may be enumerated as follows:—

(1) Ionization without dissociation . (H_2^+).
(2) Ionization with dissociation . . (H and H^+).
(3) Excitation without dissociation . (H_2^*).
(4) Excitation with dissociation . . (H and H^*).

† Hughes & Klein, Phys. Rev. (in print).

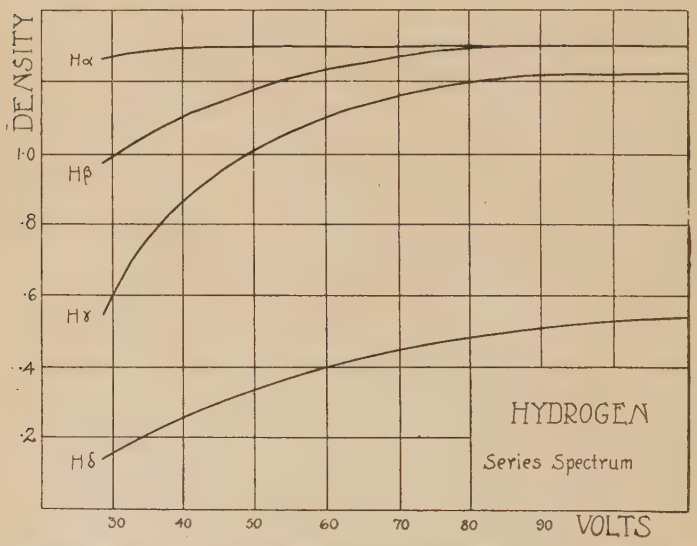
In process (1) we obtain a positive molecular ion, while in (2) we obtain an atomic ion. In (3) we have an electron removed to one of the outer orbits of the molecule, so that in falling back to its normal orbit the molecule gives out its characteristic band spectrum. In (4) the atom finds itself with an electron in one of the outer orbits, which condition enables it to emit its line spectrum. According to this way of viewing the effects of a collision between an electron and a molecule, a dissociating collision need not necessarily be accompanied by ionization ; it may equally well result in the production of one (or two) excited atoms. Smyth's results, together with those of the writer on the dissociation of hydrogen, suggest that in this gas (*a*) collision can frequently result in dissociation ; (*b*) it seldom results in the formation of an atomic ion, and therefore, according to the scheme outlined above, (*c*) the dissociation must always be accompanied by excitation of one (or both) atoms. In view of Smyth's conclusions, the ionization curve of fig. 1 must be interpreted as the number of molecular ions formed at different energies of impact, the number of atomic ions being negligible. Had Smyth's results not been available, however, the similarity in the shapes of the ionization curve and the dissociation curve in fig. 1 would have naturally been taken to indicate that the ions obtained were atomic ions accompanying dissociation, and not molecular ions.

The writer and Mr. Lowe carried out an investigation on the distribution of intensity in the spectrum of hydrogen as a function of the energy of impact between the electrons and molecules. This research *, together with another on helium †, was undertaken to find out what variations in the distribution of intensity were produced as one, and only one, variable, viz. the energy of impact of the electron, was changed. This is clearly the most fundamental variable in the excitation of the spectrum of any gas, as it is the one whose effect would persist at infinitely low pressure, when the mutual effect of atoms causing the pressure-effect would no longer be appreciable. It would also persist at infinitely small current densities, when the presence of charges in the gas in the form of electrons or ions would no longer appreciably influence the emission of radiation from any excited atom. The energy of impact of the electron on the atom is therefore to be regarded as the most fundamental variable in studies of energy distribution in the spectrum.

* Hughes & Lowe, Phys. Rev. xxi. p. 292 (1923).
† Hughes & Lowe, Proc. Roy. Soc. civ. p. 480 (1923).

In the experiments referred to, the electrons were accelerated from an oxide-coated filament through a grid into a field-free space between the grid and a "plate" at the same potential. The "plate" was so shaped that it almost completely enclosed the region in which the spectrum was excited. The pressure was sufficiently low, so that any ions produced would reach the metallic boundaries of the region and be discharged far oftener than they would recombine with electrons. Consequently we may infer that the production of the spectrum under these particular experimental conditions should be attributed almost entirely to excited, or partially ionized, molecules or atoms, and hardly at all to recombination between ions and electrons. This is a considerable simplification in the conditions of excitation of

Fig. 3.



the spectrum. Under these circumstances it is found that the intensities of the series lines increased asymptotically to a limit as the energy of impact of the electron is increased, and, moreover, the increase is greater the higher the term number of the line (fig. 3). On the other hand the intensities of the secondary lines associated with the molecule fall off rapidly with increasing energy of impact. These results would indicate that process (4) increases, while process (3) decreases, as the energy of impact of the electron is increased. It is unfortunate that the hydrogen spectrum was not studied above 100 volts energy of impact. However, up to this point there is a rough parallelism between

the dissociation curve for hydrogen and the curve for the excitation of the series lines. A close parallelism is not to be expected, for each series line is to be associated with a very special process, while the dissociation curve, on our view, would correspond to the aggregate of all such processes. Thus a dissociation of the hydrogen molecule into two atoms, in one or both of which the electron is shifted to the 5th orbit and thence falls back directly to the 2nd orbit (since Hγ is denoted by $R(1/2^2 - 1/5^2)$ in the Bohr theory), would be registered by the Hγ curve in fig. 3. The dissociation curve, on the other hand, would correspond to the sum of all possible transitions in the atoms after dissociation.

These spectroscopic results support, qualitatively at any rate, the view that process (4) occurs frequently as a result of an encounter between an electron and a molecule of hydrogen.

Whiddington * has recently published a short paper on the excitation of the hydrogen spectrum. He points out that in the laboratory one seldom gets more than four lines of the Balmer series, while in the stars over thirty lines have been found. He suggests that in the stars (1) the pressure is probably extremely low, and (2) the gas may be bombarded continuously and intensely by electrons from the interior. The first condition favours the production of many lines by reason of the comparative isolation of the radiating atom and the consequent realization of its possible outer orbits ; the second condition because of the greater opportunity of combination of the positive hydrogen ion with an electron rather than with a negative ion. He describes an experiment in which he used a tube provided with an incandescent tungsten wire cathode to provide sufficient electrons at all degrees of exhaustion. He obtained only the first four lines when his hydrogen was at 1 mm. pressure, but when he reduced the pressure to below ·001 mm., about twenty lines of the series were identified. The results of the writer and Mr. Lowe suggest an alternative explanation. According to fig. 3, when the electrons with low energies collide with molecules, the lines of higher term number are relatively much weaker than when the impact energy is high. In Whiddington's experiments at 1 mm. pressure the mean free path of the electron is only ·082 cm., and consequently it seems extremely unlikely that the electrons could pick up nearly as much energy before making collisions under these conditions as when the pressure was a thousand times

* Whiddington, Phil. Mag. xlvi. p. 605 (1923).

less. Hence it might well be that the effect observed by Whiddington is to be attributed to differences in impact energy rather than to mere change in pressure irrespective of impact energy. Whiddington remarks that increasing the current (in the experiments at 1 mm. pressure) failed to change appreciably the relative intensities of the lines. This would follow our view, since the average energy of impact would not be altered if the potential across the tube remained constant. (There is, of course, a difference between Whiddington's experiments and ours, in that in our work the emission of radiation is attributed to excitation alone, while in his work recombination occurs in addition to excitation.)

In a recent letter to 'Nature'*, Horton and Davies, in discussing certain features of Lemon's work, consider that certain changes in the appearance of the hydrogen spectrum are due to changes in the energy of the bombarding electrons, and not to changes in the intensity of the electron current apart from any changes in impact energy. This view is in agreement with the view taken by the writer and Mr. Lowe in their recent investigations on the distribution of intensities in the spectrum of hydrogen and helium, that the fundamental variable is the energy of impact of the electron on the molecule.

Summary.

1. A discussion of Smyth's investigation of the nature of the positive ion (*i. e.* whether atomic or molecular) following ionization by electron impact, and of the writer's investigation on the dissociation of hydrogen by electron impacts, points to the conclusion that dissociation of hydrogen molecules by electron impacts is accompanied much more often by excitation of the atoms than by ionization.

2. That dissociation of hydrogen molecules is frequently accompanied by excitation of the atoms receives strong support from an investigation of the distribution of energy in the spectrum of hydrogen as a function of the energy of impact of the electron.

3. Attention is drawn to the view that the fundamentally important variable in determining the character of a spectrum and, in particular, the distribution of intensity in it, is the energy of impact of the electron on the molecule.

Washington University,
Saint Louis, U.S A.,
March 22, 1924.

* Horton & Davies, 'Nature,' p. 273 (Feb. 23, 1924).

VI. *On the Equation of Long Waves in Canals of Varying Sections.* *By* N. Sen, *Ph.D.**

MANY solutions of the equation of propagation of long waves in canals of varying sections have been obtained suitable for special forms of sections. The earliest investigator on the subject was Green †, who showed that in the case of progressive waves in a canal with slowly varying sections the elevation of water is inversely proportional to the square root of the breadth and to the fourth root of the average depth at a section. Soluble cases for different types of sections have been investigated by many other writers ‡. In the present paper some special solutions, analogous to those corresponding to the motion of progressive waves, have been discussed, and the laws of variations of depth and breadth for which such types of motion can exist have been deduced, and the possibility of a wider applicability of Green's law has been indicated. It will, in fact, be found that the function $b^{\frac{1}{2}}h^{\frac{1}{4}}$ (b=breadth, h=average depth of a section) plays a very important part in the theory. The subject has not, in reality, been dealt with here as a problem in wave-motion with considerations of the proper boundary conditions, but the investigation has been limited to the discussion of certain types of solutions of the general equation with suitable values for the two arbitrary functions which correspond to the breadth and depth at any section of the canal. A similar and detailed study has been made by Wedderburn § in the case where there is only one arbitrary function in the equation, and his results correspond to motions in canals of constant breadth. We shall retain here both the arbitrary functions b and h. Any novelty in the types of solutions discussed here is not claimed, but some simple formulæ have been obtained which automatically adjust the two arbitrary functions in such a way as to lead directly to the solutions under consideration.

If η be the elevation of liquid, b and S the breadth

* Communicated by the Author.

† Green's Mathematical Papers, p. 225.

‡ Chrystal, "Some Results in the Mathematical Theory of Seiches," Proc. R. Soc. Edin. xxv. p. 328 (1904); Lamb, 'Hydrodynamics,' 3rd edition, p. 259.

§ J. Wedderburn, "On Long Waves," Amer. J. Math. p. 211 (1914).

and area respectively of any transverse section of the canal, the equation of motion is

$$\frac{\partial^2\eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(S\frac{\partial\eta}{\partial x}\right); \quad \ldots\ldots \quad (1)$$

and if h be the average depth at any section,

$$S = bh,$$

and we have

$$\frac{\partial^2\eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(bh\frac{\partial\eta}{\partial x}\right), \quad \ldots\ldots \quad (2)$$

where b and h are both functions of x. Henceforth, by depth we shall always mean the average depth over a section.

(1).

Let us assume that equation (2) has a solution of the type

$$\eta = \phi(h, b)\,\mathrm{F}.[\chi(x) \pm \sigma t], \quad \ldots\ldots \quad (3)$$

where F is an arbitrary function of its argument. We shall investigate the conditions under which this is possible. Substituting in (2), we have

$$\phi\,.\frac{\sigma^2 b}{g}\mathrm{F}'' = \mathrm{F}\,.\frac{\partial}{\partial x}(bh\,\phi') + \mathrm{F}'.\left[hb\,\chi'\phi' + \frac{\partial}{\partial x}(hb\,\phi\chi')\right] + \mathrm{F}''.\,bh\,\phi\chi'^2,$$

where the dashes in ϕ' and χ' mean differentiations with respect to x. Since F is arbitrary, we have

$$\chi'^2 = \frac{\sigma^2}{gh}, \quad \ldots\ldots \quad (4)$$

$$hb\,\phi'\chi' + \frac{\partial}{\partial x}(hb\,\phi\chi') = 0, \quad \ldots\ldots \quad (5)$$

and

$$\frac{\partial}{\partial x}(bh\,\phi') = 0. \quad \ldots\ldots \quad (6)$$

Hence

$$\chi(x) = \frac{\sigma}{\sqrt{g}}\int\frac{dx}{\sqrt{h}}. \quad \ldots\ldots \quad (4')$$

Dividing out (5) by $bh\,\phi\chi'$ and integrating, we have

$$\phi^2 hb\,\chi'(x) = \text{const.},$$

which, taken with (4), gives

$$\phi\,h^{\frac{3}{2}}h^{\frac{1}{2}} = \text{const.} \quad \ldots\ldots \quad (5')$$

This is Green's law. Equation (6) can now be written as

$$bh\frac{d}{dx}(b^{-\frac{1}{2}}h^{-\frac{1}{4}}) = \text{const.} = \alpha. \quad . \quad . \quad . \quad (6')$$

This admits of further integration. Let us now introduce a new variable given by

$$\frac{dx}{dz} = \sqrt{h},$$

so that

$$z = \frac{\sqrt{g}}{\sigma}\chi.$$

Then putting ϑ for $b^{\frac{1}{2}}h^{\frac{1}{4}}$, equation (6′) can be written as

$$\vartheta^2\frac{d}{dz}\left(\frac{1}{\vartheta}\right) = \alpha,$$

whence by integration

$$\vartheta = \text{const.} - \alpha z.$$

We have then the following relation between b, a, and χ:

$$b^{\frac{1}{2}}h^{\frac{1}{4}} = \frac{\sqrt{g}}{\sigma}(\beta - \alpha\chi), \quad . \quad . \quad . \quad . \quad . \quad (6'')$$

where α and β are any two arbitrary constants. Hence ϕ varies inversely as a linear function of χ. We have now expressed all the functions b, h, and ϕ in terms of χ. When any one of these four functions is given as a function of x, equations (4′), (5′), and (6″) determine immediately the remaining three.

From (5′) it follows that ϕ is proportional to $b^{-\frac{1}{2}}h^{-\frac{1}{4}}$, so that Green's law of amplitude holds good in this case. But this also requires the simultaneous fulfilment of equation (6′) or (6″) as an additional condition under which the law remains true. Green's restriction is that the section should vary slowly, so that the variations of h and b within the limits of a wave-length may be neglected when compared with h and b themselves. But it is evident that equation (6′) or (6″) can be satisfied (at least theoretically) under more general conditions, and Green's law of amplitude will continue to hold good. It should, of course, be admitted that for rapid variations the question of reflexion would arise, but such cases we shall exclude here.

There are two arbitrary constants α and β in (6″). $\alpha = 0$ corresponds to the case in which at every section of the

canal $hb^2 = \text{const.}$ and the function ϕ in the solution is a constant. If, now, either one of b and h is known as a function of x, the other one is given immediately by (6″) and (4′). A similar consideration also holds good for $\beta = 0$. Hence, corresponding to any given law of breadth, there are, in general, two different laws of depth for which a motion of the type (3) can exist, and conversely, when the depth is known, there are two laws of breadth answering to it. It should also be noticed that in the wave (3) any particular phase moves with velocity $(gh)^{\frac{1}{2}}$; hence when the law of variation of *breadth* is given, there are two motions possible with two different velocities of propagation. We add here certain examples.

If we take

$$hb^2 = (cx+d)^{-\frac{1}{\kappa}},$$

then we have

$$\chi = \gamma - \frac{\sigma}{\sqrt{g}\alpha} \cdot (cx+d)^{-\frac{1}{4\kappa}},$$

$$h = \alpha^2 \frac{16\kappa^2}{c^2} (cx+d)^{2+\frac{1}{2\kappa}},$$

$$b = \frac{c}{4\kappa\alpha} (cx+d)^{-\frac{3}{4\kappa}-1},$$

α and γ being constants.

In particular, putting $\kappa = -\frac{3}{4}$, we have

$$b = \text{const.}, \quad h = \frac{1}{b^2}(cx+d)^{4/3}, \quad \text{and} \quad \chi(x) = \gamma + \frac{3\sigma b}{\sqrt{g}c}(cx+d)^{1/3};$$

and putting $\kappa = -\frac{1}{4}$, we have

$$h = \text{const.}, \quad b = \frac{1}{\sqrt{h}}(cx+d)^2, \quad \text{and} \quad \chi(x) = \gamma + \frac{\sigma}{c\sqrt{gh}}(cx+d).$$

Again, we can easily show from (6″) that

if h varies as $e^{\kappa x}$, then $b \propto e^{-\frac{3}{2}\kappa x}$ or $e^{-\frac{1}{2}\kappa x}$ and $\chi \propto e^{-\frac{1}{2}\kappa x}$,

and if h varies as x^m, then $b \propto x^{2-\frac{3m}{2}}$ or $x^{-\frac{m}{2}}$ and $\chi \propto x^{1-\frac{m}{2}}$,

the second set of values of b corresponding to the case $\alpha = 0$, or $hb^2 = \text{const.}$ Similarly, for a definite value of b there will be two different values of h suitable for the propagation of the motion given by (3). The second sets of solutions (viz. $h = \text{const.}$, $b = \text{const.}$) in the first two cases considered above correspond to canals with uniform sections.

If there be a definite functional relation between h and b

at every section, we can proceed directly from (6′). Taking, for instance,

$$b = f(h),$$

and substituting in (6′) we have on integration

$$(\beta - 4\alpha x) = \int \left[\frac{\sqrt{f(h)}}{h^{\frac{1}{2}}} + 2h^{\frac{3}{2}} \frac{f'(h)}{\sqrt{f(h)}} \right] dh, \quad . \quad . \quad (7)$$

which gives the necessary relation between a and x. In particular, if b varies as h^n, then the law of depth favourable for the propagation of the waves of the present type is given by

$$h \propto (\beta - 4\alpha x)^{\frac{4}{2n+3}}.$$

The function $bh^{\frac{1}{2}}$ plays an important part in the whole theory. The value of this function can be easily calculated from the differential equation in terms of the function η. For instance, if we put

$$dz_1 = \frac{dx}{\sqrt{gh}} \quad \text{and} \quad \psi = bh^{\frac{3}{2}},$$

we can write the differential equation as

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{1}{\psi} \frac{\partial}{\partial z_1} \left(\psi \frac{\partial \eta}{\partial z_1} \right).$$

In this new form the two arbitrary functions in the original equation have been replaced by one. Introducing $\frac{\partial \eta}{\partial z_1}$ both up and down on the right-hand side, we can write the equation in the form

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial \eta}{\partial z_1} \cdot \frac{\partial}{\partial z_1} \left\{ \log \left(\psi \frac{\partial \eta}{\partial z_1} \right) \right\},$$

from which we have on integration :

$$\psi = \frac{c}{\frac{\partial \eta}{dz_1}} \cdot e^{\int \frac{\frac{\partial^2 \eta}{\partial t^2}}{\frac{\partial \eta}{\partial z_1}} dz_1 + f(t)}$$

For any given type of solution this form determines immediately the requisite value of ψ, *i. e.* of $bh^{\frac{1}{2}}$. In the present problem only those values of η can be of any interest to us which make ψ ultimately independent of t. It can be easily verified that this condition will be satisfied for a

solution of the form (3) if ϕ be so chosen that $\frac{1}{\phi}$ is a linear function of z_1, *i.e.* of χ (*cf.* equation 6″). It is to be noticed that there is only one arbitrary quantity in ψ; the constant c can be absorbed in the function of time f.

(2).

It is well known that if the width of a canal be constant at all sections while the depth varies as x, a solution of the differential equation can be obtained in the form

$$\eta = \text{const.}\, J_0(\kappa x^{1/2}) \cos \kappa\sigma t$$
$$= 2c \int_0^{\pi/2} \cos(\kappa x^{1/2} \sin\theta) \cos \kappa\sigma t \, d\theta$$
$$= c \int_0^{\pi/2} [\cos \kappa(x^{1/2} \sin\theta + \sigma t) + \cos \kappa(x^{1/2} \sin\theta - \sigma t)] \, d\theta.$$

This suggests generalization by Fourier's theorem and a solution of the form

$$\eta = \int_0^{\pi/2} [F(x^{1/2} \sin\theta + \sigma t) + F(x^{1/2} \sin\theta - \sigma t)] \, d\theta,$$

where F is an even function of its argument. This leads us to try the following integral:

$$\eta = \phi(b,h) \int_0^{\pi/2} [F\{\chi(x) \sin\theta + \sigma t\} + F\{\chi(x) \sin\theta - \sigma t\}] \, d\theta \quad \ldots \quad (8)$$

as a solution of equation (2), where F is an arbitrary even function. Substituting in the differential equation, and after a single integration by parts, we have

$$\chi'^2 = \frac{\sigma^2}{gh}, \quad \ldots \quad (9)$$

$$\frac{d}{dx}\left(bh \frac{d\phi}{dx}\right) = 0, \quad \ldots \quad (10)$$

$$\frac{d}{dx}(hb\,\phi\chi') + hb\,\chi' \frac{d\phi}{dx} - hb\,\phi \frac{\chi'^2}{\chi} = 0. \quad \ldots \quad (11)$$

From (10) we have

$$bh\frac{d\phi}{dx} = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10')$$

and the equation (11) immediately admits of the integral

$$hb\,\phi^2\frac{\chi'(x)}{\chi(x)} = \text{const.};$$

whence we have

$$\phi = \frac{c}{\sqrt{hb}}\sqrt{\frac{\chi(x)}{\chi'(x)}}, \quad . \quad . \quad . \quad . \quad . \quad (11')$$

c being a constant. Equation (10′) can now be written as

$$\left[\sqrt{\frac{1}{hb}\frac{\chi(x)}{\chi'(x)}}\right]^2\frac{d}{dx}\left[\sqrt{\frac{1}{hb}\frac{\chi(x)}{\chi'(x)}}\right] = -\kappa\frac{\chi'(x)}{\chi(x)},$$

where κ is a constant; from this we have on integration

$$bh = [\alpha+\kappa\log\chi(x)]^2\frac{\chi(x)}{\chi'(x)}, \quad . \quad . \quad . \quad . \quad (12)$$

and consequently

$$\phi = c[\alpha+\kappa\log\chi(x)]^{-1}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In this case also ϕ is expressed explicitly as a function of χ, though its dependence on h and b is not so simple as in (5″). When any one of the functions b, h, χ, and ϕ is known as a function of x, equations (9), (12), and (13) determine the values of the other three in such a way that equation (2) has the integral (8) for a solution. One of these four functions we can choose always arbitrarily. The appearance of two arbitrary constants α and κ in (12) indicates that here also, as in the previous case, corresponding to any value of either one of h and b there exist two different values of the other function such that a motion of the present type may become possible. For the particular value

$$\chi(x) = \sqrt{x}$$

we have

$$h = \frac{4\sigma^2}{g}x, \quad bh = 2x[\alpha + \tfrac{1}{2}\kappa \log x]^2$$

and

$$\eta = c(\alpha + \tfrac{1}{2}\kappa \log x)^{-1}\int_0^{\pi/2}[\mathrm{F}(x^{1/2}\sin\theta + \sigma t) + \mathrm{F}(x^{1/2}\sin\theta - \sigma t)]\,d\theta.$$

If in these results we put $\kappa = 0$, we pass on to the case of constant breadth and of the depth varying as x. Secondly, if we put

$$h = \text{const.} \quad \text{and} \quad \chi'(x) = \frac{\sigma}{\sqrt{gh}} = \alpha,$$

then

$$\chi(x) = \alpha x + \beta \quad \text{and} \quad b = \left(\frac{x}{h} + \frac{\beta}{\alpha h}\right)\{c + \kappa \log(\alpha x + \beta)\}^2$$

and

$$\eta = c'\{c + \kappa \log(\alpha x + \beta)\}^{-1}\int_0^{\pi/2}[\mathrm{F}(\overline{\alpha x + \beta}\sin\theta + \sigma t) + \mathrm{F}(\overline{\alpha x + \beta}\sin\theta - \sigma t)]\,d\theta.$$

In this result, if we put $\kappa = 0$, then we have

$$b \propto (\alpha x + \beta).$$

Again, putting $\chi(x) = e^{mx}$, we have

$$h = \frac{\sigma^2}{gm^2}e^{-2mx}, \quad b = \frac{gm}{\sigma^2}e^{2mx}(\alpha + \kappa m x)^2$$

and

$$\eta = c(\alpha + \kappa m x)^{-1}\int_0^{\pi/2}[\mathrm{F}(e^{mx}\sin\theta + \sigma t) + \mathrm{F}(e^{mx}\sin\theta - \sigma t)]\,d\theta.$$

(3).

The solution found in § 1 can be generalized by assuming that η depends not only on F, but also on the differential coefficients of this function. Following the suggestion of Wedderburn *, let us put

$$\eta = \phi_0(h, b)\,\mathrm{F}[\chi(x) + \sigma t] + \phi_1(h, b)\,\mathrm{F}'[\chi(x) + \sigma t] + \phi_2(h, b)\,\mathrm{F}''[\chi(x) + \sigma t] + \ldots + \phi_n(h, b)\,\mathrm{F}^n[\chi(x) + \sigma t]. \quad \ldots \quad (14)$$

* *Loc. cit*

Substituting in equation (2) and equating coefficients of $F, F', \ldots$, we have

$$\left.\begin{aligned}
\frac{\partial}{\partial x}(bh\phi_0') &= 0,\\
bh\chi'\phi_0' + \frac{\partial}{\partial x}(bh\chi'\phi_0 + bh\phi_1') &= 0,\\
bh\chi'\phi_1' + \frac{\partial}{\partial x}(bh\phi_1\chi_1' + bh\phi_2') &= \left(\frac{\sigma^2}{g} - h\chi'^2\right)b\phi_0,\\
bh\chi'\phi_2' + \frac{\partial}{\partial x}(bh\phi_2\chi' + bh\phi_3') &= \left(\frac{\sigma^2}{g} - h\chi'^2\right)b\phi_1,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
bh\chi'\phi_n' + \frac{\partial}{\partial x}(bh\phi_n\chi') &= \left(\frac{\sigma^2}{g} - h\chi'^2\right)b\phi_{n-1},\\
0 &= \left(\frac{\sigma^2}{g} - h\chi'^2\right)b\phi_n.
\end{aligned}\right\} \quad (15)$$

The first and the last two equations give immediately on integration

$$bh\phi_0' = \text{const.}, \quad \phi_n = c_n b^{-\frac{1}{2}} h^{-\frac{1}{4}}, \quad \text{and} \quad \chi' = \frac{\sigma}{\sqrt{gh}},$$

whereas the relation between any two consecutive ϕ's is given by the remaining equations of (15). Writing the general equation in the form

$$2bh\chi'\phi'_{r-1} + \phi_{r-1}\frac{\partial}{\partial x}(bh\chi') + \frac{\partial}{\partial x}(bh\phi'_r) = 0,$$

and expressing χ' in terms of h and making the following substitutions:

$$bh^{1/2} = \vartheta, \quad dz = \frac{dx}{\sqrt{h}},$$

we can also write it as follows:

$$2\vartheta\frac{\partial\phi_{r-1}}{\partial z} + \phi_{r-1}\frac{\partial\vartheta}{\partial z} + \frac{\sqrt{g}}{\sigma}\frac{\partial}{\partial z}\left(\vartheta\frac{\partial\phi_r}{\partial z}\right) = 0. \quad . \quad (15')$$

This is the general equation to determine ϕ_r, the two extreme equations having the integrals

$$\phi_n = c_n\vartheta^{-\frac{1}{2}} \quad \text{and} \quad \vartheta\frac{\partial\phi_0}{\partial z} = \text{const.}$$

In its general form this equation is difficult to handle, but particular solutions may sometimes be obtained easily. For instance, if we assume $\vartheta = z^m$ and consequently $\phi_n = c_n z^{-\frac{m}{2}}$, we can easily show that ϕ_{n-r} is of the form

$$\phi_{n-r} = c_{n-r} z^{-\frac{m}{2}-r},$$

where the constant c_{n-r} is given by the recurrence formula

$$2rc_{n-r} + \frac{g}{\sqrt{\sigma}}\left(\frac{m}{2}-r\right)\left(\frac{m}{2}+r-1\right)c_{n-r+1} = 0.$$

Now we have to choose m in such a way that the equation

$$\vartheta \frac{\partial \phi_0}{\partial z} = \text{const.}$$

is automatically satisfied. Here we distinguish between two cases. If we take ϕ_0 ($\neq 0$) itself to be constant, then we must also take

$$\frac{m}{2}+n=0, \text{ i.e. } m=-2n,$$

and

$$\vartheta = z^{-2n}, \quad \phi_{n-r} = c_{n-r} z^{n-r};$$

or if we take ϕ_0 ($\neq 0$) to be different from constant, we have to put

$$\frac{m}{2}-n-1=0, \text{ i.e. } m=2(n+1),$$

and

$$\vartheta = z^{2(n+1)}, \quad \phi_{n-r} = c_{n-r} z^{-(n+r+1)}.$$

The corresponding values of b and h can be easily found.

We can get a completely soluble case if we take η to consist of two terms only, *e. g.*

$$\eta = \phi_0 \mathrm{F} + \phi_1 \mathrm{F}'.$$

We have, as before,

$$\chi'^2 = \frac{\sigma^2}{gh}, \quad \phi_1 = c_1 b^{-\frac{1}{2}} h^{-\frac{1}{2}}, \quad hb\phi_0' = \text{const.} = \alpha,$$

and

$$\frac{\partial}{\partial x}(bh\phi_0\chi') + \alpha\chi' + \frac{\partial}{\partial x}(bh\phi_1') = 0;$$

integrating, we have

$$bh\phi_0\chi' + \alpha\chi + bh\phi_1' = \beta \text{ (const.)}.$$

Substituting the value of ϕ_0 obtained from this equation

in the third of the above equations and introducing a slightly different notation,

$$\Theta = b^{-\frac{1}{2}}h^{-\frac{1}{4}},$$

we have ultimately

$$\frac{d}{dz}\left[-\frac{c_1\sqrt{g}}{\sigma}\frac{d\Theta}{dz}+\left(\beta\frac{\sqrt{g}}{\sigma}-\alpha z\right)\Theta^2\right] = \alpha\Theta^2.$$

This is a non-linear equation and has no simple general solution. A particular solution is, however, given by

$$\Theta = -\frac{c_1}{\beta}\cdot\frac{1}{z}.$$

But we get a completely soluble case if we put $\alpha=0$, which corresponds to $\phi_0=\text{const.}=c_0$ and to the solution

$$\eta = c_0\mathrm{F}\left(\frac{\sigma}{\sqrt{g}}z+\sigma t\right)+c_1b^{-\frac{1}{2}}h^{-\frac{1}{4}}\mathrm{F}'\left(\frac{\sigma}{\sqrt{g}}z+\sigma t\right).$$

It can be shown that in this case the above equation has solutions of the form

$$\Theta = \kappa_1\sigma'\frac{e^{-\sigma'\kappa_1\kappa_2(z+\kappa_3)}+e^{\sigma'\kappa_1\kappa_2(z+\kappa_3)}}{e^{-\sigma'\kappa_1\kappa_2(z+\kappa_3)}-e^{\sigma'\kappa_1\kappa_2(z+\kappa_3)}},$$

where $\sigma'=\frac{\sigma^{1/2}}{g^{1/4}}$ and κ_1, κ_2, κ_3 are three constants of which κ_1 and κ_2 are connected by the relation $\frac{c_0}{c_1}=\kappa_1^2\kappa_2$; or

$$\Theta = \kappa_1\sigma'\tan\sigma'\kappa_1(\kappa_3-\kappa_2 z),$$

depending on the signs of the constants in the above differential equation (when $\beta\neq0$). The simplest case, however, arises when we put β also $=0$. The following illustrations are then perhaps interesting:—

$h=\text{const.}$,

$$b=\frac{1}{\sqrt{h}x^2} \quad\text{and}\quad \eta = c_0\mathrm{F}\left(\frac{x}{\sqrt{gh}}+t\right)+c_1x\mathrm{F}'\left(\frac{x}{\sqrt{gh}}+t\right);$$

$b=\text{const.}$,

$$h=\frac{x^4}{b^2} \quad\text{and}\quad \eta = c_0\mathrm{F}\left(-\frac{b}{\sqrt{g}x}+t\right)+\frac{c_1}{x}\mathrm{F}'\left(-\frac{b}{\sqrt{g}x}+t\right).$$

In such cases we can, of course, make use of the solution

of the general equation (15′) given above. In the above examples the solutions correspond to $\beta=0$; other solutions corresponding to the constant values of h and b can be easily found from the general solutions given above.

(4).

In the manner of the preceding section we can find solutions of equation (2) in terms of the integral of § 2 and its differential coefficients. We discuss here a case which is simple and completely integrable. We suggest the following solution:

$$\eta=\phi(h,b)\frac{\partial}{\partial x}\int_0^{\pi/2}[\mathrm{F}\{\chi(x)\sin\theta+\sigma t\}+\mathrm{F}\{\chi(x)\sin\theta-\sigma t\}]\,d\theta, \quad \ldots \quad (16)$$

where F is an even function of its argument. Substituting in the differential equation and proceeding in the usual way, we get the following conditions:

$$\left.\begin{aligned} \chi'^2 &= \frac{\sigma^2}{gh}, \\ \phi^2 hb\frac{\chi'^3(x)}{\chi(x)} &= \text{const.}, \\ \frac{d}{dx}\left[bh\frac{d}{dx}(\phi\chi')\right]+bh\phi\frac{\chi'^3}{\chi^2} &= 0. \end{aligned}\right\} \quad \ldots \quad (17)$$

Eliminating ϕ and h from the third equation with the help of the two preceding ones, we have

$$\frac{d}{dx}\left[\frac{b}{\chi'^2}\cdot\frac{d}{dx}\left(\sqrt{\frac{\chi\chi'}{b}}\right)\right]+\frac{b}{\chi^2}\sqrt{\frac{\chi\chi'}{b}}=0.$$

Writing this equation in the form

$$\frac{\chi}{\chi'}\frac{d}{dx}\left[\frac{b}{\chi\chi'}\cdot\frac{\chi}{\chi'}\frac{d}{dx}\left(\sqrt{\frac{\chi\chi'}{b}}\right)\right]+\sqrt{\frac{b}{\chi\chi'}}=0,$$

and making the following substitutions:

$$\sqrt{\frac{\chi\chi'}{b}}=\xi \quad \text{and} \quad \frac{\chi}{\chi' dx}=\frac{\chi}{d\chi}=\frac{1}{dy}, \quad \text{i.e.} \quad y=\log\chi,$$

we have after simplification

$$\frac{d^2}{dy^2}\left(\frac{1}{\xi}\right)-\frac{1}{\xi}=0.$$

This has the solutions

$$\frac{1}{\xi} = e^{y} \text{ and } e^{-y};$$

whence we have

$$\sqrt{\frac{\chi\chi'}{b}} = \chi \text{ and } \frac{1}{\chi}.$$

In the first case we have

$$\left.\begin{aligned} b &= \frac{\chi'}{\chi} \quad \text{and} \quad h = \frac{\sigma^2}{g}\cdot\frac{1}{\chi'^2}, \\ \text{and in the second} \\ b &= \chi^3\chi' \end{aligned}\right\} \quad . \quad . \quad (18)$$

with the same value for h. We can now easily find

$$\phi = \frac{e\sqrt{g}}{\sigma}\sqrt{\frac{\chi}{b\chi'}}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If we choose χ arbitrarily we shall have two corresponding values for b, and if we choose b arbitrarily we shall have two corresponding values for χ and consequently two values for h also.

The following cases are perhaps interesting :—

$$\chi(x) = x^n, \quad h = \frac{\sigma^2}{gn^2}x^{2(1-n)}, \quad b = nx^{4n-1} \text{ or } \frac{n}{x},$$

and

$$\eta = \kappa x^{1-2n}\frac{\partial}{\partial x}\int_0^{\pi/2}[\mathrm{F}\{x^n\sin\theta+\sigma t\}+\mathrm{F}\{x^n\sin\theta-\sigma t\}]\,d\theta$$

$$\text{or} \qquad \kappa x\frac{\partial}{\partial x}\int_0^{\pi/2}[\mathrm{F}\{x^n\sin\theta+\sigma t\}+\ .\ .\ .\ .\ .\ .\ .\ .\ ]\,d\theta$$

according as b is equal to nx^{4n-1} or nx^{-1} respectively (κ being a constant).

Putting $n=1$, we have the case of constant depth, and

$$h = \frac{\sigma^2}{g}, \quad b = x^3 \text{ or } \frac{1}{x},$$

and

$$\eta = \frac{\kappa}{x}\frac{\partial}{\partial x}\int_0^{\pi/2}[\mathrm{F}(x\sin\theta+\sigma t)+\ .\ .\ .\ ]\,d\theta$$

$$\text{or} \qquad \kappa x\frac{\partial}{\partial x}\int_0^{\pi/2}[\mathrm{F}(x\sin\theta+\sigma t)+\ .\ .\ .\ ]\,d\theta$$

respectively.

Again, for constant values of b we have the solutions :

$$b = \text{const.}, \quad h = \frac{8\sigma^2}{g\sqrt{b}}x^{3/2} \text{ or } \frac{\sigma^2}{gb^2}e^{-2bx},$$

and

$$\eta = \kappa x^{\frac{1}{2}} \frac{\partial}{\partial x} \int_0^{\pi/2} [\mathrm{F}(\sqrt[4]{4b}x^{\frac{1}{4}} \sin\theta + \sigma t) + \ldots] \, d\theta$$

or

$$\kappa \frac{\partial}{\partial x} \int_0^{\pi/2} [\mathrm{F}(e^{bx} \sin\theta + \sigma t) + \ldots] \, d\theta,$$

corresponding to the first or second value of h. The last solution has been given by Wedderburn.

Again, putting

$$\chi(x) = e^{\lambda x},$$

we have

$$h = \frac{\sigma^2}{g\lambda^2} e^{-2\lambda x}, \quad b = \lambda \text{ or } \lambda e^{4\lambda x}.$$

The solution corresponding to the constant value of b is the one already given above ; that corresponding to the second value of b is

$$\eta = \kappa e^{-2\lambda x} \frac{\partial}{\partial x} \int_0^{\pi/2} [\mathrm{F}(e^{\lambda x} \sin\theta + \sigma t) + \mathrm{F}(e^{\lambda x} \sin\theta - \sigma t)] \, d\theta.$$

Note.—The results of § 1 can be deduced more simply if the equation of motion (2) be slightly transformed as follows. Using the same notation as in § 1, we can write (2) as

$$\frac{\partial^2(\mathfrak{H}\eta)}{\partial t^2} = \frac{g}{\mathfrak{H}} \frac{\partial}{\partial z} \left(\mathfrak{H}^2 \frac{\partial \eta}{\partial z} \right),$$

which is the same as

$$\frac{\partial^2(\mathfrak{H}\eta)}{\partial t^2} = g \left(\frac{\partial^2(\mathfrak{H}\eta)}{\partial z^2} - \eta \frac{\partial^2 \eta}{\partial z^2} \right).$$

From this it at once follows that $\eta\mathfrak{H}$ satisfies the usual wave equation, and is equivalent to an arbitrary function of the form $\mathrm{F}(z \pm \sqrt{g}t)$ if $\mathfrak{H}$ satisfies the equation

$$\frac{\partial^2 \mathfrak{H}}{\partial z^2} = 0,$$

or $\mathfrak{H}$ is a linear function of z, which is the same result as equation (6). A clue to the other types of solutions of equation (2) can be obtained if we take the relation between $\mathfrak{H}$ and z to be given by

$$\frac{1}{\mathfrak{H}} \frac{\partial^2 \mathfrak{H}}{\partial z^2} = \phi(z),$$

$\phi(z)$ being a given function of z. Then the whole problem of wave motion in canals reduces to the solution of an equation of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial z^2} - \phi(z) u.$$

VII. *The Diameters of the Atoms of the Inert Gases.*

By J. E. Calthrop, *B.A.(Cantab.), B.Sc.(Lond.)**.

VALUES for the radii of the atoms of the inert gases have been given by Davey † from a consideration of the packing radii of alkali and halogen ions in crystal lattices. These are much greater than those given by Prof. W. L. Bragg, and in most cases even greater than the estimates of Prof. Rankine from viscosity data and from Chapman's formula:—

$$\sigma^2 = \frac{0{\cdot}491\zeta\overline{V}}{\sqrt{2}n\pi\eta\left(1+\frac{c}{T}\right)},$$

in which

σ = molecular diameter,
ρ = density,
$\overline{V}$ = mean molecular velocity,
n = number of molecules per c.c.,
η = viscosity,
T = absolute temperature,
c = Sutherland's constant.

Davey finds that Rankine's results are related to his by the formula $(R-0{\cdot}35)\sqrt{2}=D$, where R is the Rankine radius and D his own. Further, he suggests that Chapman's formula could be modified empirically to give results which agree with his crystal method, though he can find no theoretical basis for such a modification.

In a recent paper ‡ he states a new law: "The ratio of the radii of any two simple atoms belonging to the same vertical column in the periodic table is the same as the ratio of the radii of any other two simple atoms in the same rows of the table, and in a common vertical column." This enables him to predict the diameters of the atoms of certain elements, including helium, which had not been deduced previously from crystal data.

His value for the diameter of the Helium atom is 2·06 Å, which is near Rankine's value of 2·16 Å.

It is of interest to find what value is obtained from the refractivity, for we have recently § pointed out that the refractivities and the Bragg values of the atomic diameters of the inert gases are related by the equation $(\mu-1)=1{\cdot}23N\sigma^3$,

* Communicated by the Author.
† Physical Review, xxii. p. 211 (1923).
‡ Physical Review, xxiii. p. 318 (1924).
§ Phil. Mag. xlvii. p. 772 (1924).

where $(\mu-1)$ is the refractivity of the gas, N is the number of atoms per c.c. and is taken as $2{\cdot}705\times10^{19}$, and σ is the atomic diameter.

If we may include helium in the same group, and use the refractivity $34{\cdot}9\times10^{-6}$ given by Cuthbertson, we obtain a diameter of 1·02 Å. This is in good agreement with the earlier estimate of M. Zierucci * from the temperature of fusion. He gave the equation

$$T_f=79D_A-81,$$

where T_f is the absolute temperature of fusion, and D_A is the atomic diameter in Ångström units. T_f is near 1° for helium, and the diameter obtained is 1·04 Å.

We may find how far the Davey values obey an equation of the type $(\mu-1)=KN\sigma^3$, in which K is a constant which holds for the Bragg values. In the following table K is calculated both from the data of Davey and Bragg, except that we consider 1·02 Å to be the "Bragg" value for helium.

		Davey.		Bragg.	
Element.	Refractivity $\times10^6$.	σ.	K.	σ.	K.
He	34·9	2·06 Å	0·15	1·02 Å	1·23
Ne	68·7	2·52	0·16	1·30	1·16
Ar	283·7	3·14	0·34	2·05	1·24
Kr	427·3	3·42	0·39	2·35	1·24
Xe	702·0	3·94	0·42	2·70	1·35

Thus the Davey estimates do not appear to give so simple a relation between the refractivities and diameters as do those of Bragg.

It is noticeable, however, that the differences between the two sets of diameters are fairly constant. Consequently, if we take the mean of these differences, 1·13 Å, and employ a formula

$$(\mu-1)=KN(\sigma-1{\cdot}13)^3$$

where K is now the Davey diameter, we get the following values of K:—

Element.	$K=\frac{(\mu-1)}{N(\sigma-1{\cdot}13)^3}$.
He	1·60
Ne	0·95
Ar	1·29
Kr	1·32
Xe	1·17

* *Nuovo Cimento*, xxii. p. 189 (1921).

The mean value of K is 1·27, and thus we might suggest $(\mu-1)=1{\cdot}27N(\sigma-1{\cdot}13)^3$ as the equation of the Davey values. On the whole, the evidence of refractivity appears to be in favour of the Bragg diameters, as there seems no reason why the refractivity should be uninfluenced by diameters less than 1·13 Å.

East London College,
May 6th, 1924.

VIII. *On the Chance of an Electron being Ejected Photoelectrically from an Atom by X-rays.* *By* G. E. M. Jauncey, *D.Sc., Assistant Professor of Physics, Washington University, U.S.A.**

1. *Introduction.*

THE total mass absorption coefficient of X-rays in matter is made up of two parts thus:

$$\mu/\rho = \tau/\rho + \sigma/\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where τ/ρ is the true mass absorption coefficient and σ/ρ is the mass scattering coefficient, which for light elements and soft X-rays is of the order 0·2 gram^{-1}. From the true absorption coefficient, the true absorption coefficient per atom, or the true atomic absorption coefficient, may be found by multiplying τ/ρ by the atomic weight and dividing by Avogadro's number. Owen † and others have found the empirical relation that the true atomic absorption coefficient, which we shall represent by τ/ν, is equal to $k\mathrm{N}^4\lambda^3$, where N is the atomic number of the absorbing substance, λ is the wave-length of the X-rays, and k is a constant. The quantity k, however, is only constant within certain limits. When a K, L, or M etc. absorption limit is passed the value of k changes abruptly.

The energy which is lost from the primary beam due to true absorption goes into the production of photoelectrons and K, L, M etc. characteristic radiations. However, it will be assumed that the production of characteristic radiations is a secondary process, and that the primary effect of the X-rays is to produce photoelectrons. A photoelectron cannot be ejected from an atom unless the energy of the

* Communicated by the Author.
† E. A. Owen, Proc. Roy. Soc., A. xciv. p. 522 (1918).

X-ray quantum is greater than the binding energy of the electron. This binding energy is determined by the K, L, M etc. absorption limits. It so happens that according to Moseley's law the K absorption limit wave-length is very nearly inversely proportional to the square of the atomic number. Now M. de Broglie * has shown that, when the wave-length of the X-rays is short enough to excite the K radiations of a substance, the number of photoelectrons ejected from the K energy level is greatly in excess of the number ejected from the L and M energy levels. It seems, then, that so long as the K radiations are excited the greater part of the energy lost from the primary beam goes into the production of photoelectrons ejected from the K level. Since the binding energy of the electrons in the K level is proportional to N^2 approximately, it seems that the true atomic absorption coefficient varies as the square of the binding energy of the electrons in the K level, provided that the K radiations are excited.

2. *Probability of Photoelectric Emission.*

The above considerations led the writer to make the hypothesis that the chance of an electron being ejected photoelectrically from one of the K, L, or M etc. levels is proportional to the square of the binding energy of the electron in the level from which it is ejected, provided that the wave-length is short enough to eject the electron from the level and provided that the wave-length of the X-rays is kept constant. Let λ_K, λ_L, and λ_M be the absorption limit wave-lengths for the K, L, and M radiations respectively from a given absorbing material. The binding energies of the electrons in the levels are then inversely proportional to these quantities respectively. Each atom has n_K, n_L, etc. electrons in the K, L, etc. levels respectively. On the writer's hypothesis the chance of a K electron being ejected from an atom is proportional to n_K/λ_K^2, the chance of an L electron being ejected is proportional to n_L/λ_L^2, and so on for the electrons in the other levels. Assuming that the chance of an atom being doubly ionized is negligible, we then have that the chance of any one of the K, L, or M etc. electrons being ejected from an atom is proportional to

$$P = n_K/\lambda_K^2 + n_L/\lambda_L^2 + n_M/\lambda_M^2 + \ldots \quad (2)$$

Keeping the wave-length constant and varying the absorber,

* M. de Broglie, *Jour. d. Phys. et Rad.* ii. p. 265 (1921).

the atomic absorption coefficient should be proportional to P. If the wave-length λ of the primary X-rays is longer than λ_K, the K term drops out and so on. Since the true absorption coefficient also varies as λ^3, we may say that the chance of an electron being ejected from an atom is proportional to $\lambda^3 P$. The true atomic absorption coefficient being supposed to be due to the ejection of photoelectrons, this quantity should be given by

$$\tau/\nu = G\lambda^3(n_K/\lambda_K^2 + n_L/\lambda_L^2 + \ldots), \quad \ldots \quad (3)$$

where G is a constant. The relation (3) has been obtained theoretically by A. H. Compton * on the classical theory and by L. de Broglie † on the quantum theory. However, in both cases the theoretical approach involves so many other factors that it is difficult to see what assumption it is that results in the $n_K/\lambda_K^2 +$ etc. term in the absorption equation (3). On the writer's view this term merely means that the probability of photoelectric emission is proportional to the square of the binding energy. The writer therefore suggests that Owen's law be replaced by the following empirical law:

> "The chance of an electron being ejected photoelectrically from an atom by X-rays is proportional to the cube of the wave-length of the X-rays and to the square of the binding energy of the electron to the atom, provided that the wave-length is short enough to eject the electron."

This law will now be supported by showing from experimental evidence that the true atomic absorption coefficient is proportional to $\lambda^3 P$. The values of λ_K, λ_L, λ_M for various elements are given in Table I. These are averages of values given by Foote and Mohler ‡, Hughes §, Kurth ‖, and Sommerfeld ¶. The L and M absorption limits have a fine structure, but in the table only the average wave-lengths are given. Some of the values have been obtained by interpolation.

* A. H. Compton, Bull. Nat. Res. Council, xx. p. 39.
† L. de Broglie, *Jour. de Phys. et Rad.* iii. p. 33 (1922).
‡ Foote and Mohler, Phys. Rev. xix. p. 434 (1922).
§ A. L. Hughes, Phil. Mag. xliii. p. 145 (1922).
‖ E. H. Kurth, Phys. Rev. xviii. p. 461 (1921).
¶ A. Sommerfeld, *Atombau und Spektrallinien*, pp. 227–229

TABLE I.

Absorption Limits in Ångström Units.

Element.	Atomic Number.	λ_K.	λ_L.	λ_M.
C	6	46·2	368	—
Al	13	7·95	150	326
Fe	26	1·74	16·3	54·3
Cu	29	1·38	12·3	41·6
Mo	42	·618	4·8	18·6
Ag	47	·485	3·5	15·0
Pb	82	·141	·85	4·5

In Table II. are shown the values of n_K, n_L, n_M, n_K/λ_K^2, n_L/λ_L^2, and n_M/λ_M^2 for the various elements, λ_K, λ_L, and λ_M being in Ångström units.

TABLE II.

Element.	n_K.	n_L.	n_M.	n_K/λ_K^2.	n_L/λ_L^2.	n_M/λ_M^2.
C	2	4	0	·00094	·00003	—
Al	2	8	3	·0316	·00036	·00003
Fe	2	8	16	·66	·0302	·0055
Cu	2	8	18	1·058	·0532	·0105
Mo	2	8	18	5·24	·349	·052
Ag	2	8	18	8·52	·65	·08
Pb	2	8	18	100·5	11·1	·9

The values of G obtained by dividing the experimental value of the true atomic absorption coefficient by $\lambda^3 P$ are shown in the sixth column of Tables III., IV., and V. These values of G are for the case where λ, λ_K, λ_L, etc. are measured in Ångström units. In these tables the true mass absorption coefficients are obtained by subtracting the mass scattering coefficient from the respective values of the total absorption coefficients given by A. H. Compton in a table in the Bulletin of the National Research Council on Secondary Radiations produced by X-rays. The mass scattering coefficient used in calculating the true absorption coefficient is not that given by Thomson's theory *, viz., $0{\cdot}4019 \times p/w$ gram^{-1}, where p is the number of electrons per atom and w the atomic weight. The scattering coefficient is

* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 325.

somewhat doubtful. A. H. Compton * and the writer † have shown in their quantum theories of scattering that the scattering coefficient is $1/(1+2a)$ times the Thomson coefficient, where $a=h/mc\lambda$. Thus for $\lambda=0{\cdot}35$ Å.U.,

$$a=\cdot024/\cdot35=\cdot069,$$

and the mass scattering coefficient for carbon is

$$0{\cdot}201/1{\cdot}138=0{\cdot}176.$$

The total mass absorption coefficient in carbon is given in Compton's table as 0·219, leaving 0·043 as the true absorption coefficient.

TABLE III.

Wave-length . . . 0·35 A.U.

Characteristic Radiations Excited.	Element.	τ/ρ Experimental.	$\tau/\nu\times10^{23}$ Experimental.	P.	$\frac{\tau/\nu}{\lambda^3 P}\times10^{20}$.
K, L	C	·043	·084	·00097	2·01
K, L, M	Al.....	·56	2·49	·032	1·80
K, L, M	Fe.....	4·59	42·4	·695	1·42
K, L, M	Cu ...	6·77	70·6	1·122	1·46
K, L, M	Mo ...	18·9	299·0	5·64	1·24
K, L, M	Ag.....	27·0	481·0	9·25	1·21
L, M	Pb.....	22·4	765·0	12·0	1·48

TABLE IV.

Wave-length . . . 0·70 Å.U.

Characteristic Radiations Excited.	Element.	τ/ρ Experimental.	$\tau/\nu\times10^{23}$ Experimental.	P.	$\frac{\tau/\nu}{\lambda^3 P}\times10^{20}$.
K, L	C	·34	·67	·00097	2·01
K, L, M	Al.....	4·76	21·1	·032	1·92
K, L, M	Fe.....	37·7	349·0	·695	1·46
K, L, M	Cu ...	52·5	547	1·122	1·42
L, M	Mo ...	18·5	298	0·401	2·16
L, M	Ag.....	31·5	561	0·73	2·24
L, M	Pb.....	163·6	5540	12·0	1·34

* A. H. Compton, Phys. Rev. xxi. p. 483 (1923).

† G. E. M. Jauncey, Phys. Rev. xxii. p. 233 (1923), and xxiii. p. 313 (1924).

TABLE V.

Wave-length . . . 1·0 Å.U.

Characteristic Radiations Excited.	Element.	τ/ρ Experimental.	$\tau/\nu \times 10^{23}$ Experimental.	P.	$\frac{\tau/\nu}{\lambda^3 P} \times 10^{20}$
K, L	C	1·07	2·12	·00097	2·18
K, L, M	Al	13·6	60·5	·032	1·89
K, L, M	Fe	90·0	832	·695	1·20
K, L, M	Cu	146·0	1518	1·122	1·35
L, M	Mo ...	53·0	840	·401	2·10
L, M	Ag	87·0	1550	·73	2·12

Examining values of G given in the last columns of Tables III., IV., and V., we see that G is fairly constant, varying only by a factor of 1·85 from the smallest to the largest value, whereas in Table III. τ/ν varies by a factor of about 9000. Furthermore, it is seen that the value of G remains approximately constant when an absorption limit is passed, as, for instance, in Table III. when we pass from silver to lead. The lack in the constancy of G may possibly be due to several causes. First, the absorption wave-length limits are not known to a high degree of accuracy, especially in the case of the L and M absorption limits. Then, too, the N, O, P, etc. limits have not been considered. Second, the total absorption coefficients are somewhat uncertain, as in the case of the elements of low atomic number very great purity is necessary and in the case of the elements of high atomic number only very small thicknesses can be used. Third, each of the L, M, etc. absorption limits has a fine structure. Instead of n_L/λ_L^2 we should write the sum of three terms, there being three L absorption limits. To make this correction it would be necessary to know the numbers of electrons per atom in the L_1, L_2, and L_3 energy levels. Fourth, there is great uncertainty in the value assigned to the scattering coefficient. A. H. Compton * has measured the spectrum of scattered X-rays and has found that part of the scattered radiation suffers a change of wave-length while part of it does not. It seems probable that the scattering coefficient of the unmodified radiation would be given by Thomson's value. Instead, therefore, of subtracting $0·2/(1+2a)$ from the total absorption, we should subtract some quantity between this and 0·2 in order to obtain the true absorption coefficient. It seems possible that if all

* A. H. Compton, Phys. Rev. xxii. p. 409 (1923).

these sources of error could be eliminated G might be found to be a constant. The average of all the values of G in Tables III., IV., and V. is $1{\cdot}71 \times 10^{-20}$.

Further, it is found experimentally that the absorption "jump" for a given absorbing substance when the X-rays are decreased in wave-length past the K absorption limit decreases with increasing atomic number. Recently Allen * and Warburton and Richtmeyer † have shown that the ratio of the K + L + M + ... absorption to the L + M + ... absorption at the K absorption limit is about 7, but decreases for high atomic numbers. On the present view this ratio should be

$$\frac{n_K/\lambda_K^2 + n_L/\lambda_L^2 + n_M/\lambda_M^2 + \ldots}{n_L/\lambda_L^2 + n_M/\lambda_M^2 + \ldots} \quad . \quad . \quad . \quad . \quad (4)$$

Referring to Table II., it is seen that this ratio decreases from 19·5 for Fe to 9·4 for Pb, which is in the general direction of the experimental results.

3. *Discussion.*

The writer believes that considerable experimental evidence has been quoted to support the proposition that the chance of an electron being ejected photoelectrically from an atom (*i. e.* such that $h\nu = mv^2/2 + W$) is proportional to the cube of the wave-length and to the square of the energy of binding, provided that the wave-length is short enough to eject the electron. The writer believes that this way of interpreting the experimental results may have importance from the theoretical point of view. It seems as though the X-ray quantum possesses an affinity for the electron which increases as the binding energy increases. It would seem then that the chance of an X-ray quantum giving up its entire energy to an electron in an electron vapour would be zero, since in this case there would be no binding energy. Hence there is probably no photoelectric effect in an electron vapour. In such a vapour the X-ray quantum should only be scattered. This scattering would take place with change of wave-length of the scattered quantum, part of the energy of the original quantum going into the production of a "recoil" electron as shown by A. H. Compton ‡. The writer has recently published a Corpuscular Quantum Theory of the Scattering

* S. J. M. Allen, Phys. Rev. xxiii. p. 291 (1924).
† Warburton and Richtmeyer, Phys. Rev. xxiii. p. 291 (1924).
‡ A. H. Compton, Phys. Rev. xxi. p. 483 (1923).

of X-rays *, and has shown in a recent letter to 'Nature' † that if an electron has a radius of $4{\cdot}57 \times 10^{-13}$ cm., and if the X-rays are supposed to be corpuscles which are scattered when they hit an electron, the mass scattering coefficient would be $0{\cdot}2$ gram^{-1}, which is very nearly that found experimentally. It seems that the photoelectric effect of X-rays is altogether a different process from the scattering of X-rays, as all the collisions of the X-ray corpuscles with electrons are accounted for by the observed value of the scattering coefficient. It seems possible that the photoelectric effect takes place only when the X-ray quantum or corpuscle passes between the electron and the nucleus. In fact it seems that the presence of the nucleus is a necessary factor in the photoelectric effect. It is interesting to speculate as to why it is that on the quantum passing between the electron and the nucleus the chance of the electron being ejected varies as the square of the binding energy. Bubb ‡ in a recent letter to 'Nature,' introduces the idea of a vector quantum. He speaks of the quantum "exploding" in a direction perpendicular to its line of flight. In his view the nucleus provides the seat of the reaction when the electron is ejected.

It would be very interesting to test whether anything like the empirical law proposed in this paper holds in the case of light. It is perhaps worth while noting that Hughes and Klein § have shown that the chance of ionization per collision increases as the binding energy decreases for helium, neon, and argon, if the speed of the ionizing electron is kept constant. This implies that the chance of an electron being ejected by collision with another electron increases as the binding energy decreases and in this way varies in the opposite direction to that when the electrons are ejected photoelectrically by X-rays.

Washington University,
St. Louis, Mo., U.S.A.
March 22, 1924.

* G. E. M. Jauncey, Phys. Rev. xxii. p. 233 (1923), and xxiii. p. 313 (1924).
† G. E. M. Jauncey, 'Nature,' cxiii. p. 196 (1924).
‡ F. W. Bubb, 'Nature,' cxiii. p. 237 (1924).
§ Hughes and Klein, Phys. Rev., in print.

IX. *On the Collision of Spherical Bodies of Unequal Diameters and Densities at Very Low Velocities. By* D. B. Deodhar, *M.Sc., Reader in Physics, Lucknow University, Lucknow, India* *.

Introduction.

ASSUMING certain conditions, Hertz † has given an interesting discussion about the phenomena which occur when two balls impinge on each other. He has found out that the radius of the circle of pressure, which is the boundary of the surface of contact, varies as the $\frac{3}{2}$ power of the radius of the impinging sphere, and that the duration of impact, that is the time during which the bodies are in contact, is very large compared with the gravest period of vibration of either of the balls. Schneebeli ‡, Lafay §, and Northrup ‖ have done some work to compare experimentally the theory of impact put forth by Hertz. In all the cases investigated by these workers they found that their experimental results were not in agreement with the theoretical deductions.

It is well known that the total kinetic energy before impact cannot be equal to that after impact unless the impact is perfectly elastic; in all other cases there will be dissipation of energy in various forms. Hertz has shown that the maximum pressure increases with the relative velocity, being proportional to the $\frac{2}{5}$ power of the velocity; while it is independent of the size of the balls. And the greater the velocity the more will the pressure exceed the elastic limit, and there will be a dissipation of energy. This dissipation was not contemplated by Hertz. Lord Rayleigh ¶ did some work on the impact of balls, and showed that the proportion of translational energy transferred into the energy of vibration is very small. The dissipation of energy due to sound has been discussed by Banerji **, who has shown that the sound produced by the impact of spheres is due to the impulse communicated to the atmosphere by the sudden reversal of the motion of the spheres. The dissipation of the energy may largely be

* Communicated by the Author.
† Hertz, 'Miscellaneous Papers' (English translation), pp. 146–147.
‡ *Rep. d. Phys.* xxii. (1886).
§ *Comptes Rendus*, 1900.
‖ Frank. Inst. Jour. 1911.
¶ Phil. Mag. xi. p. 283 (1906).
** Phil. Mag. July 1916, Jan. 1918.

ascribed to the failure of the condition of perfect elasticity during impact. When we know that even small spheres travelling with a speed as low as one centimetre per second develop a pressure which is far greater than what perfect elastic conditions would allow, the dissipation of energy must inevitably result; and if the bodies be plastic a great deformation in the area of contact can be easily noticed. Again, it should also be expected that the relationship between the velocities of impinging bodies before and after impact, namely the coefficient of restitution, must vary in value, becoming less and less as the relative velocity increases. This has been observed recently by Vincent *. Similarly, if the velocity of approach be made very small, the coefficient of restitution should increase and tend towards unity. Very recently Raman † has made observations upon impact of bodies at very low velocities, and has shown for two or three kinds of materials that the coefficient of restitution does really approach unity. In all his experiments Raman worked with balls of the same material and equal diameters. The present author thought it worth while to make experimental investigations as to how the coefficient of restitution is affected if the impinging balls be of different materials and unequal diameters; it was also thought interesting to find the influence of the change in the medium in which the balls were colliding.

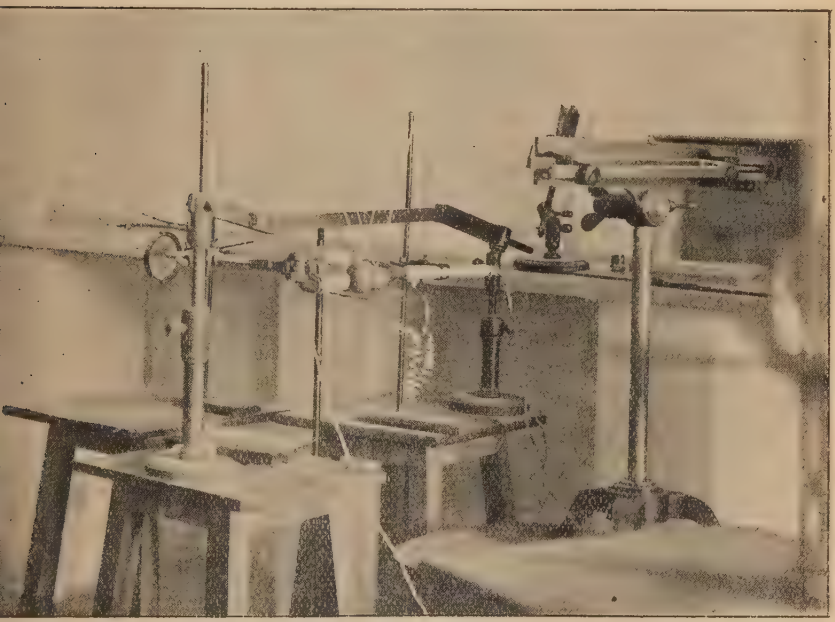
Experimental Technique.

Experimental arrangements were in essentials the same as those of Raman ‡; and as the observations were to be made for very low velocities, the relative velocities were recorded and measured by a photographic method. In order to obtain a slow and steady motion, the balls were suspended by a bifilar suspension in such a way that their height was eighteen feet from the ceiling. A horizontal slit was illuminated by a Pointolite lamp of one hundred candle-power through a condensing lens, and the image of this slit was focussed by a convergent lens on the point of contact of the two balls. Consequently no light could be seen on the other side of the balls unless they were held apart. This image was further focussed upon the slit of the camera by an additional convex lens suitably placed. A metre scale placed horizontally behind the balls ensured that on being released they fell through equal distances. A very small

* Camb. Phil. Soc. Proc. 1900.
† Phys. Rev. vol. xii. (1918).
‡ Phys. Rev. vol. xii. (1918).

and light iron piece was attached to the remote side of each of the balls, and two small electromagnets were arranged so that on closing the circuit the balls could be held apart through any required distance, and on breaking the circuit they could approach each other and collide. (See photograph of the arrangement in fig. 1.) The camera consisted of a closed long wooden box, on the back of which a photographic plate is made to slide vertically downwards

Fig. 1.



through a side groove. The motion of the photographic plate is regulated by means of a lead counterweight attached to the end of a string passing over a frictionless pulley. By properly adjusting the counterweight, the plate can be dragged up slowly and fairly uniformly past the slit. The switch controlling the electromagnets is so arranged that the experimenter can allow the counterweight to fall and then break off the current in such a way that the approach and rebounding of the balls take place while the photographic plate is moving past the slit of the camera. In this way two wedges of light are produced on the photographic plate corresponding to approach and rebound. The

gap between these wedges indicates the duration of impact—that is, the time during which the balls were in contact with each other. The ratio between the widths of the wedges measured at equal distances from the apexes is obviously equal to "*e*," the coefficient of restitution. In order to have a time-scale in these measurements, a large tuning-fork of 70 vibrations per second was mounted in front of the camera in such a way that a beam of light proceeding from another Pointolite lamp and made converging by means of a fine circular aperture and a lens, and reflected from a light mirror attached to the prong of this fork, did not obstruct the beam of light coming direct from the impinging balls. By this arrangement a reflected image of the circular aperture could be produced upon a second slit provided for in the camera. Thus the vibration curves of the tuning-fork, as well as the light wedges due to impact, could be simultaneously photographed upon the same plate; and duration of impact and other time-intervals could be measured by referring to the vibration curves of the fork.

For taking observations on the impact in water, the impinging balls were immersed in a glass basin of water placed on the top of a screw-motion table. The current in the electromagnets was adjusted so as to keep the balls in contact with the sides of the basin containing water. By selecting basins of larger or smaller diameters, the balls could be made to fall through any desired distance.

Observations and Discussion.

Experiments were carried out with wooden and brass balls, which were specially turned in the laboratory workshop ; the wood selected was the best Indian Shisham (sp. gr. = 1·13), which could sink in water. The wooden balls were 1·75 in. in diameter, while the brass balls were 2 in. and 1·5 in. in diameter respectively. The surfaces of these balls were polished to an equal extent as far as possible. After each impact, the surfaces taking part in actual collision were inspected to see if there was any permanent deformation in the way of small indentations ; but nothing of the kind was detected for any of the sets of observations made. All the observations show that when the velocities are made extremely small, the coefficient of restitution rapidly tends to approach the value unity and actually becomes equal to 1, in all cases irrespective of any change either in the material of the colliding balls or in their diameters. It appears that Hodg-

kinson's * formula for the coefficient of restitution between two dissimilar bodies, which is

$$e_{AB} = \frac{e_{AA}/q_1 + e_{BB}/q_2}{\frac{1}{q_1} + \frac{1}{q_2}}$$

(where e_{AB} = coefficient of restitution between the bodies A and B, and q_1, q_2 are Young's modulii for the two bodies), does not hold good when the velocities are extremely small. The duration of impact appears to be greater than what is to be expected from the theory put forth by Hertz. According to Hertz †, the duration of impact for two spheres of equal radii and equal density is given by the relation

$$T = 2{\cdot}9432\,R\sqrt[5]{\frac{25\pi^2 s^2(1-\sigma^2)^2}{8(V-U)q^2}},$$

where R = radius of sphere, s = density of the material, V = velocity of approach of one sphere, U = velocity of the second sphere, q and σ are Young's modulus and Poisson's ratio for the substance of which the spheres are made. At small velocities the duration of impact was observed to be independent of the diameters and materials of the spheres; and what is more interesting to note is that, other conditions being the same, the duration is greater in water than in air. This has been found invariably for all observations. The duration of impact could be very well measured by comparing the gap between the wedges of light on the photographic negative with the vibration curve of the tuning-fork. It appears that the general formula for the duration of impact should be modified by introducing an additional term corresponding to the density of the medium to account for the larger duration of impact in water. The results of several observations are given below in a graphical form. Figures 2 *a* and 2 *b* represent the case of brass balls 1·5 in. in diameter impinging in air and water respectively, while diagrams 3 *a*, 3 *b* and 4 *a*, 4 *b* represent respectively the cases of wood and brass, and brass balls of unequal diameters. In all these figures the coefficient of restitution is shown along the axis of Y, and the velocity of impact is plotted along the axis of X.

* British Association Reports, 1834.

† 'Miscellaneous Papers' (English translation).

All these figures show that when the velocities are made extremely small there is practically no effect due to variation in diameter and material upon the change in the value of the coefficient of restitution. It appears that, in addition to the dissipation of energy in sound and heat in the act of collision, there must be dissipation of energy in some kind

Brass Balls: 1·5 in. diameter each.

Fig. 2 *a*.—Air.

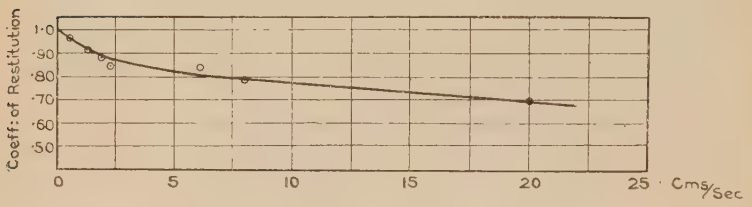
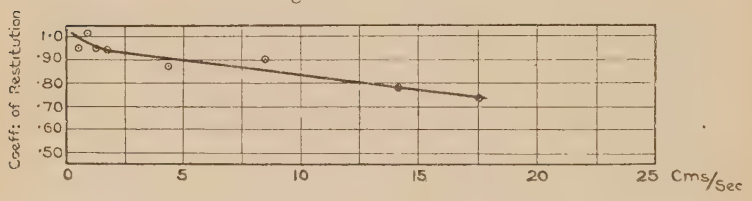


Fig. 2 *b*.—Water.



Brass Balls: 1·5 and 2·0 in. diameter.

Fig. 3 *a*.—Air.

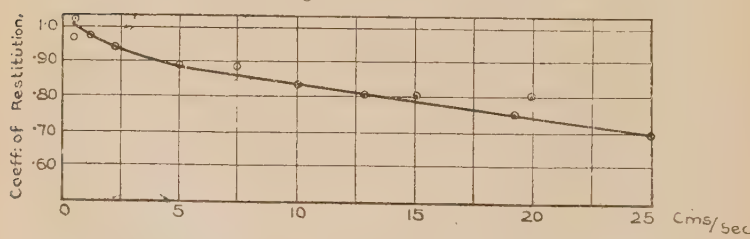
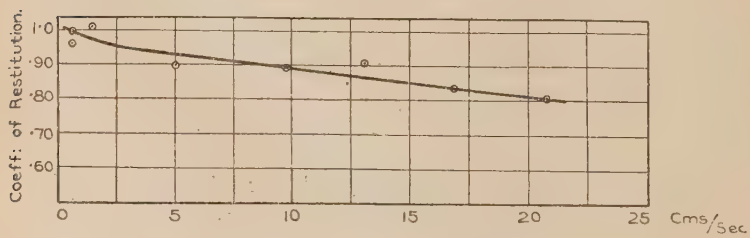


Fig. 3 *b*.—Water.



Wooden Ball: 1·75 in. diam. and Brass Ball: 1·5 in. diam.

Fig. 4 *a*.—Air.

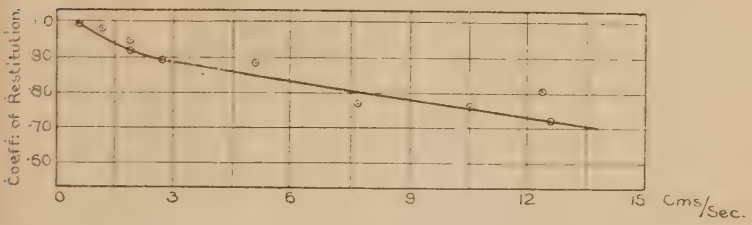
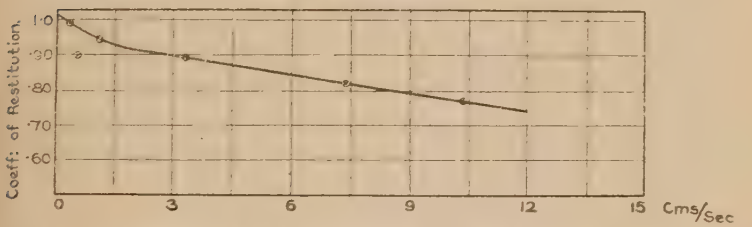


Fig. 4 *b*.—Water.



of displacement of the molecules in the surface-layer of the ball-material, owing to which the coefficient of restitution increases when the velocity is decreased, and diminishes when the velocity is increased. Since, according to Hertz, it is known that the pressure exerted upon ordinary spheres of one-inch radius colliding with a velocity of 1 cm. per second and also the simultaneous maximum pressure at the centre of impact per unit area is 73 kilograms/mm.^{2}, this pressure should produce deformation sufficiently great to be easily detected; but no such deformation was visible in the cases observed.

Now the question arises as to what happens to such an immense energy developed in the act of collision. Sound, heat, deformation, and vibration of the balls as a whole appear to be the different forms in which the energy dissipates itself. But remembering that the amount of sound energy produced is due to the reacting impulse given to the surrounding medium, and knowing that a small temperature change in the body is not enough to account for the whole energy-loss, one is inclined to think that very probably a portion of the stresses produced in the act of collision is spent in displacing the molecules in the surface-layer of the impinging balls; and that the change in the coefficient of restitution is due to such a surface change. This displacement becomes very minute when the velocity of approach is minimal, and it is only in this extreme case that the values

of the kinetic energy before and after impact are exactly equal to each other. Therefore, in all cases of collisions except those where the velocities are extremely small so as to make $e=1$, the coefficient of restitution should play a great part in the dissipation of energy.

Thinking that it would be interesting to make microscopic observations of the surfaces of impinging bodies when the velocities are so small as not to show any apparent deformation, arrangement was made to take photomicrographs of these surfaces. In two or three cases which were thus observed it was distinctly seen that the photographs of surfaces after impact differed much from those before impact. But with a view to arrive at a decisive conclusion regarding my surface-layer hypothesis, it is hoped to conduct further scrutinizing observations after introducing more refinements in the experimental conditions. The problem is certainly most interesting, but at the same time extremely difficult, as the original surface may be disturbed owing to oxidation as well as during the very process of polishing.

Summary.

1. This paper deals with impacts of spherical balls of unequal diameters and densities investigated at very low speeds.
2. Impacts are taken in air as well as in water, and the manner in which the coefficient of restitution changes is studied photographically.
3. It is found that when the velocities are extremely small, changes in diameters and densities as well as a change in the medium in which collision takes place do not produce any appreciable effect upon the coefficient of restitution.
4. The duration of impact in water is invariably larger than that in air, and this fact calls for a modification in Hertz's duration formula.
5. It appears that during the process of impact molecular displacement takes place in the surface-layer, and that the coefficient of restitution is connected with dissipation of energy.
6. It is hoped to continue the photomicrographic study further by introducing additional refinements in the experimental conditions.

In conclusion the author takes this opportunity to express his best thanks to Professor Dr. Wali Mohammad, of the University, for his very helpful and encouraging suggestions during the progress of the work.

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X. *Wave-Power Transmission.* *By* E. A. Barclay-Smith, *B.A.* (*Cantab.*), *Lieutenant R.E.**

THE Constantinesco system of wave-power transmission has been successfully applied to the synchronization of the firing of machine-gun bullets from an aeroplane and to the Dorman Long boring machine.

In his book 'Sonics' the inventor investigates this form of power transmission by means of the "j" method. By this method he is enabled to take leakage and friction into account. He obtains a series of equations which show remarkable similarity to those found in the investigation of long-distance telephone-lines.

The following is an attempt to expound the phenomena occurring in wave-power transmission and to formulate an easy method of investigating them from first principles. Furthermore, an indication is given of the line of investigation to be followed when the data are such as will be met with in practice.

The essential feature of wave-power transmission is that power is transmitted to a distance by means of a fluid contained in a pipe. Pressure-waves are generated in the fluid at one end of the pipe by means of a reciprocating plunger. The plunger is operated by a crank or its equivalent, with the result that its motion is not simple-harmonic. As in most cases, however, the length of the connecting rod is great in comparison with the length of the crank, the plunger may be considered without appreciable error to have simple-harmonic motion.

At the other end of the pipe power is transmitted to a similar plunger by the pressure-waves in the liquid. The pipe may have several branches with a plunger at the end of each, in order that power may be received by several different machines.

In the event of any machine being out of use and thus not requiring any power, it may be disconnected from its plunger which will then be left to reciprocate without absorbing power (neglecting frictional losses), or the plunger may be locked. Every plunger will of course be provided with a flywheel or other arrangement to carry it over the dead points.

It is first necessary to investigate the happenings in a single pipe full of fluid when a plunger starts to reciprocate at one end.

* Communicated by Prof. G. B. Jeffery, M.A., D.Sc.

In the following consideration the effects of friction and leakage in the pipe are neglected *. The pressures referred to in this paper are those in excess of the pressure due to the static head of fluid necessary to keep the pipe full and to prevent cavitation taking place.

Let p = the instantaneous fluid pressure in lb. per sq. inch.

K = coefficient of compressibility of the fluid in lb. per sq. inch †.

ρ = density of the fluid in lb. per cubic inch.

v = particle velocity in ins. per second.

g = acceleration due to gravity in ins. per sec. per sec.

From a consideration of the motion of a particle of the fluid in the pipe it can easily be shown that if a single pressure-wave be travelling along a pipe it travels with velocity

$$\sqrt{\frac{\mathrm{K}g}{\rho}} \text{ ins. per second,}$$

and that

$$p = \sqrt{\frac{\mathrm{K}\rho}{g}} \times v,$$

where v is the instantaneous velocity of the particles of the fluid in the direction in which the wave is travelling.

When the pressure-wave meets an obstruction part of it may continue to advance through the obstruction and thence be transmitted to a machine or to a continuum of the fluid on the further side, part of it will be reflected backwards as another pressure-wave not necessarily in phase with the original wave. This reflected wave when it reaches the generator will again be reflected in part, and this process will go on indefinitely.

Accompanying each of these pressure-waves will be a similar wave of particle velocity ; the shape of the velocity-wave will be similar to the pressure-wave and at every point will obey the law

$$p = \sqrt{\frac{\mathrm{K}\rho}{g}} \times v.$$

It is important to remember that the direction of the particle velocity corresponds with that in which the wave is travelling.

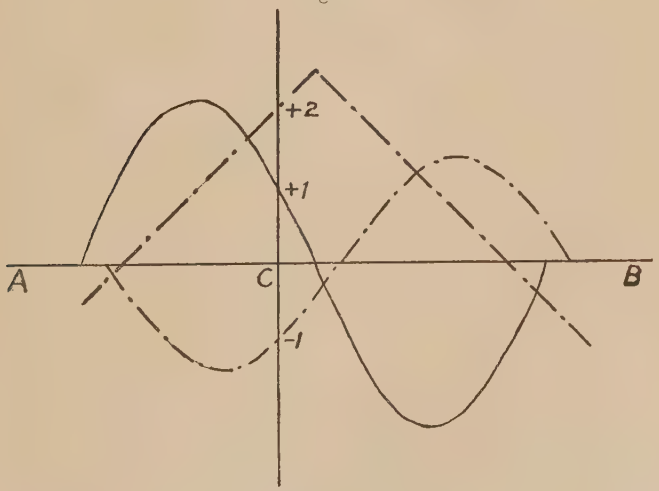
* For a full consideration of their effects *vide* 'Sonics,' p. 46 *et seq.*

† In most cases the effect of expansion of the pipe need not be considered. Its effect would be slightly to alter the apparent value of K.

Thus when the plunger starts to reciprocate there will be created in the pipe a series of pressure-waves each accompanied by its corresponding particle-velocity wave and travelling to and fro along the pipe.

At any given point at any given instant of time the pressure is equal to the *sum* of the pressures due to the pressure-waves passing that point at that instant. The particle velocity, however, is in each case * in the direction in which the wave is travelling; thus the direction of the particle velocities due to waves travelling in one direction is opposite to the direction of the particle velocities due to waves travelling in the reverse direction. Hence the resultant particle velocity will be equal to the *difference* between the sum of the velocities due to the velocity-waves travelling in one direction and the sum of the velocities due to the velocity-waves travelling in the opposite direction.

Fig. 1.

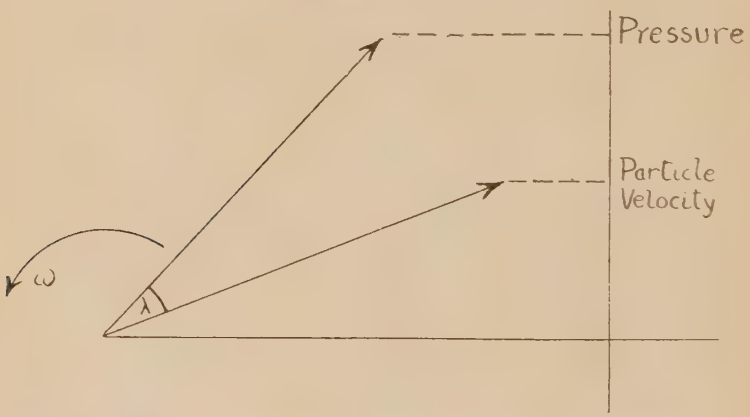


Thus if in fig. 1 the chain dotted lines represent two pressure-waves travelling along the pipe in the direction AB and the continuous line represents one travelling in the direction BA at the moment under consideration, the pressure at C will be equal to $(2-1+1)$ lb. per sq. in., while the particle velocity will be equal to $(2-1-1)\times\sqrt{\frac{g}{K\rho}}$ ins./second, and it will be in the direction AB. This rule applies whatever the shapes of the waves.

* *I. e.* "in the case of each separate individual wave."

If the waves are sinoidal and all of the same frequency, the waves in each direction can be added together to form a single sinoidal wave. The pressure at any point will vary sinoidally with time. The particle velocity at that point will also vary sinoidally with time but will not necessarily be in phase with the pressure. Hence the pressure and particle velocity at some particular point can be represented vectorially, *i. e.* the actual quantities of the pressure and particle velocities respectively are represented by the projections of their vectors as they rotate with angular velocity ω (fig. 2); the wave period, that is the time of one complete oscillation of the generator plunger, being $\frac{2\pi}{\omega}$.

Fig. 2.



If the vectors representing the pressure and the particle velocity at any given point are known, it is possible to analyse them into two sinoidal waves travelling in opposite directions by the following construction.

At the point O (fig. 3) let $\overline{OP}$ be the pressure vector and let $\overline{OQ}$ be the vector for particle velocity in the direction AB. The length OQ is the actual particle velocity $\times \sqrt{\frac{K\rho}{g}}$ to the same scale as OP.

Join PQ and bisect it at R.

Join OR and draw $\overline{OS}$ equal and parallel to QR.

Then $\overline{OR}$ will represent both the pressure and particle velocity of the component wave travelling in the direction AB, and $\overline{OS}$ will represent the pressure and particle velocity

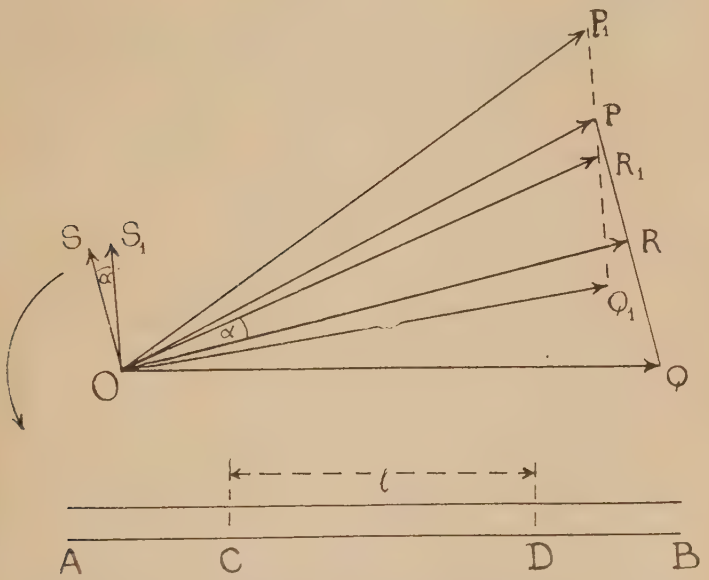
of the component wave travelling in the direction BA to the same scales as OP and OQ,

since $\overline{OR}+\overline{OS}=\overline{OP}$

and $\overline{OR}-\overline{OS}=\overline{OQ}$.

It must be remembered that the direction of the particle velocity represented by $\overline{OR}$ is AB, while the direction of the particle velocity represented by $\overline{OS}$ is BA.

Fig. 3.



Now the wave travelling in the direction AB reaches a point D distant l from C in $l\div\sqrt{\frac{K\bar{g}}{\rho}}$ seconds *after* it has reached C, while the wave travelling in the direction BA reaches D in $l\div\sqrt{\frac{K\bar{g}}{\rho}}$ seconds *before* it reaches C. Thus to find conditions at D it is necessary to turn the vector $\overline{OR}$ *forwards*, *i. e.* counter-clockwise, through an angle α and the vector $\overline{OS}$ *backwards*, *i. e.* clockwise, through the same angle where

$$\alpha=\omega l\times\sqrt{\frac{\rho}{K\bar{g}}}.$$

Let OR_1, OS_1 be the new positions of the vectors $\overline{OR}$, $\overline{OS}$. Then $\overline{OP_1}$ the sum of $\overline{OR_1}$ and $\overline{OS_1}$ will represent the pressure at D, and $\overline{OQ_1}=\overline{OR_1}-\overline{OS_1}$ will represent the particle velocity in the direction AB at D.

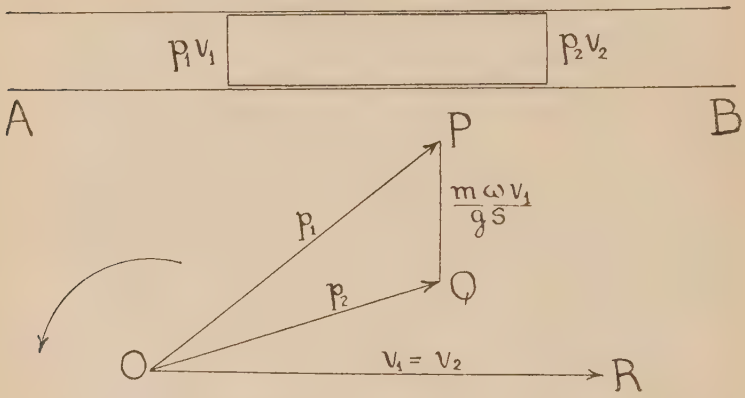
Thus if the conditions obtaining at any point are known those obtaining at any other point can be ascertained.

Inertias and Hydraulic Capacities.

An inertia acts as would a heavy metal cylinder placed in the pipe.

The particle velocity will be identical at either end of the inertia but the pressure at the two ends will differ since force must be supplied to give it the particle motion.

Fig. 4.



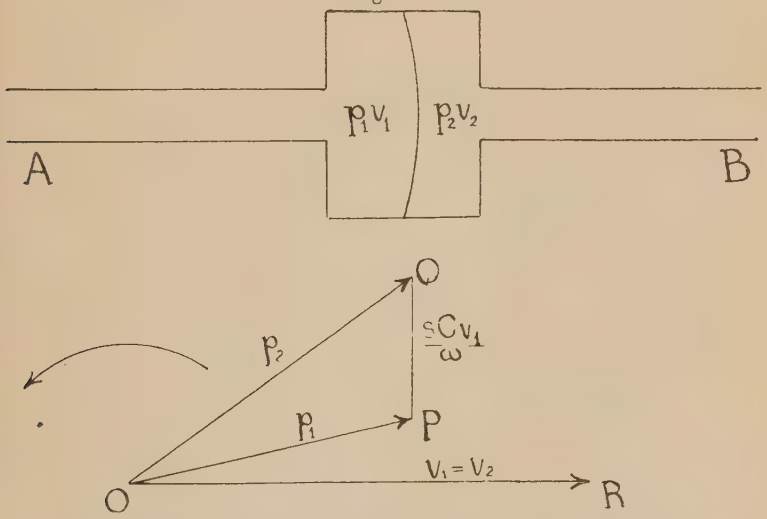
Let p_1, v_1 be the pressure and particle velocity respectively at one end and p_2, v_2 be the pressure and particle velocity respectively at the other end of the inertia, then $v_1=v_2$ in magnitude and phase. Let s be the sectional area of the pipe in square inches and m the mass of the inertia in lb. The maximum particle acceleration equals $\omega \times$ the maximum particle velocity and the acceleration vector leads the velocity vector by 90°, both the particle acceleration and the particle velocity being considered to be in the same direction.

Thus if $\overline{OR}$ (fig. 4) represents the particle velocity in the direction AB and $\overline{OP}$ represents the pressure p_1, then $\overline{OQ}$ will represent the pressure p_2, where QP is perpendicular to OR and represents $\frac{m\omega v_1}{gs}$ the pressure required to give the inertia the particle motion.

Hydraulic capacities may be of two kinds :—(1) That which acts as would a rubber diaphragm placed in the pipe, where the difference of pressure on the two sides varies as the displacement. This form is sometimes termed a condenser. (2) That which acts as would a closed vsssel containing gas placed in direct communication with the pipe, where the amount of fluid entering would vary directly as the amount of the pressure.

Taking case (1) and using the same notation as before, $v_1 = v_2$ (fig. 5) in magnitude and phase; the maximum displacement $= \frac{s}{\omega} \times$ the maximum particle velocity and the displacement vector lags 90° behind the velocity vector.

Fig. 5.



Thus if $\overline{OR}$ (fig. 5) represents the particle velocity in the direction AB and $\overline{OP}$ represents the pressure p_1, then $\overline{OQ}$ will represent the pressure p_2 where QP is perpendicular to OR and represents $\frac{sCv_1}{\omega}$ where C, a constant for the condenser, is the difference in pressure in lb. per square inch caused by the displacement of one cubic inch of fluid.

It will be noticed that if a condenser is placed in close approximation to an inertia in the pipe, and

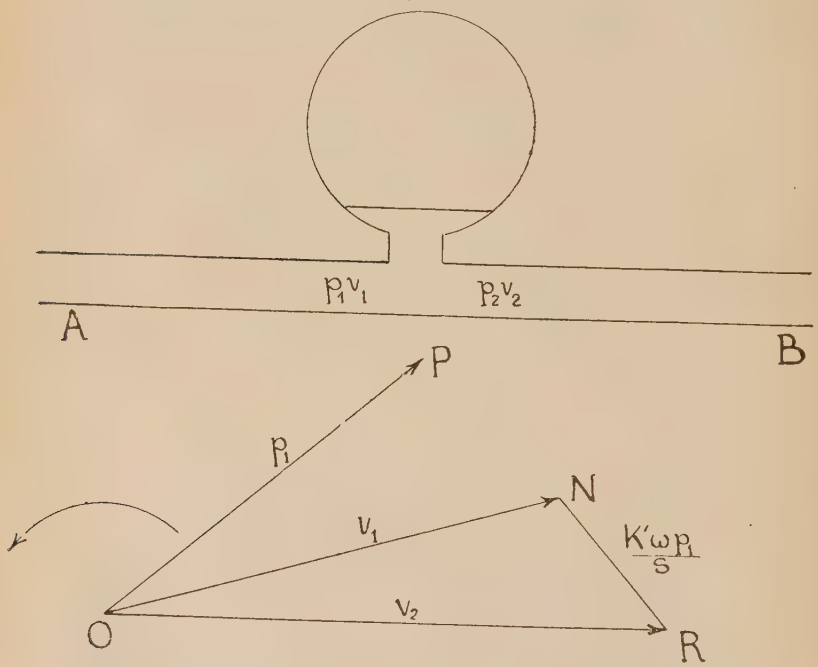
$$\frac{m\omega v_1}{sg} = \frac{sCv_1}{\omega} \quad \text{or} \quad \omega^2 = \frac{s^2C}{g},$$

the two effects neutralize one another.

Take case (2) using the same notation.

In this case $p_1 = p_2$ (fig. 6) in magnitude and phase, but v_2 differs from v_1 by the differential of the amount of fluid entering the vessel divided by s. The amount of fluid entering the vessel $= K'p_1$ cubic ins. where K', a constant for the capacity, is equal to the number of cubic inches of fluid which will enter the vessel with an increase of pressure of 1 lb. per sq. inch ; the differential of this amount will be $K'\omega p_1$, and in phase its vector will lead the pressure vector by 90°.

Fig. 6.



Thus if $\overline{OP}$ (fig. 6) represents the pressure and $\overline{ON}$ represents v_1, then $\overline{OR}$ will represent v_2 where RN is perpendicular to $\overline{OP}$ and represents $\frac{K'\omega p_1}{s}$, the particle velocities being considered to be in the direction AB.

From the above consideration it will be seen that if conditions at one point in the pipe are known it is possible to find conditions at another point in spite of the fact that an inertia or hydraulic capacity may intervene between them.

This may be done by the following process. In the first place find the component forward and backward waves at the known point, swing the vectors counter-clockwise or clockwise as the case may be through the angles necessary to find the conditions on the near side of the inertia or hydraulic capacity. The waves may now be combined and the conditions found on the far side by the addition to the pressure vector or to the velocity vector of the appropriate vector perpendicular to the velocity vector or to the pressure vector as the case may be. The conditions on the far side are to be analysed into forward and backward waves and the vectors representing the waves swung through the angle appropriate for the investigation of the conditions at the specified point. On recombining the waves the conditions required will be found.

Motors and Generators.

Although there are many devices for starting motors from rest, every motor and generator in a wave-power transmission system is provided with a plunger reciprocating synchronously with the waves in the transmitting fluid. Two or more generators might be connected to the same pipe, but in such an arrangement their reciprocations must be synchronous. Several motors may receive power from the same pipe. The pipe may vary in size and may include inertias, condensers, and transformers, but in every case the pistons of the motors must reciprocate at the same rate as the pistons of the generators, though not necessarily in phase with them. It is assumed that the generators maintain a constant rate whatever the load.

The following data are available for determining the conditions at any point in the pipe. Since the piston displacement of the cylinders of each generator and motor are known, the particle velocities at the ends of the pipe can be calculated for any particular rate of oscillation. The power absorbed by each motor must be known since this can be varied at will. The simplest case, namely that of a generator and motor at either end of a uniform pipe, will now be considered.

Let AB be the pipe of length l having the generator at A and the motor at B. Three conditions may occur :— (*a*) The piston of the motor may be fixed. (*b*) The piston of the motor may be oscillating but absorbing no power. (*c*) The motor may be absorbing power P inch-lb. per sec. Let p_A, v_A, p_B, v_B be the pressures and particle velocities at

the generator and motor respectively; v_A and v_B both being in the direction AB.

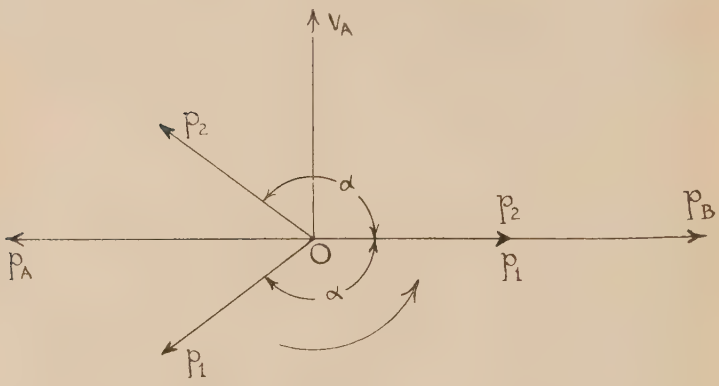
Then in every case v_A and v_B are known.

(*a*) In this case $v_B = 0$, hence the two component waves must be equal in magnitude and phase.

Let p_1 (fig. 7) be the maximum pressure of the forward wave, and p_2 that of the backward wave.

Then $$p_1 + p_2 = p_B \text{ and } p_1 = p_2.$$

Fig. 7.



In order to find the conditions at the generator turn the vector representing the forward wave backwards, *i. e.* clockwise, and the vector representing the backward wave forwards, *i. e.* counter-clockwise, in each case through an angle

$$\alpha = \omega l \times \sqrt{\frac{\rho}{\mathrm{K}g}},$$

the particle velocity at A will then be

$$2p_1 \sin\alpha \div \sqrt{\frac{\mathrm{K}\rho}{g}},$$

and this is known to be equal to v_A,

also $$p_A = 2p_1 \cos\alpha\ ;$$

thus $$p_B = v_A \operatorname{cosec}\alpha \sqrt{\frac{\mathrm{K}\rho}{g}} \text{ and } p_A = v_A \cot\alpha \sqrt{\frac{\mathrm{K}\rho}{g}}.$$

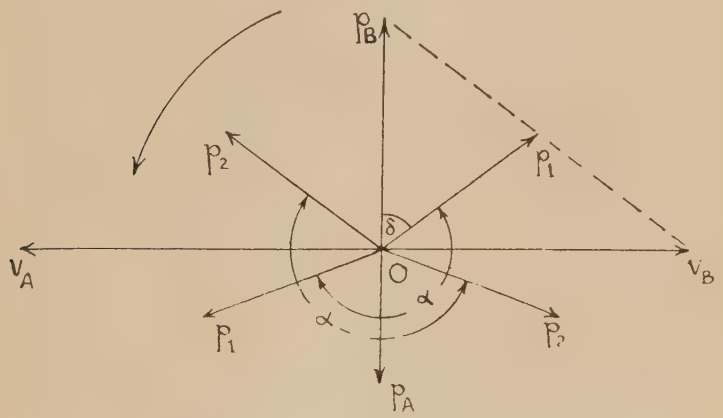
It will be noticed that if α is a multiple of π, that is

$$l = \frac{n\pi}{\omega} \sqrt{\frac{\mathrm{K}g}{\rho}},$$

the pressures at A and B would tend to become infinite. The time of one oscillation $= \frac{2\pi}{\omega}$ and the wave velocity $= \sqrt{\frac{Kg}{\rho}}$; hence $\frac{\pi}{\omega}\sqrt{\frac{Kg}{\rho}} =$ half a wave-length. Therefore if l is nearly equal to some multiple of half the wave-length, an inertia or hydraulic capacity must be inserted into the pipe in order to modify the conditions and thus prevent the bursting of the pipe.

(*b*) In this case v_B (fig. 8) is known, and as no power is absorbed the vector representing p_B must be at right angles to that representing v_B.

Fig. 8.



Analysing into component waves it will be found that $p_1 = p_2$. Let δ be the angle at which p_1 and p_2 are inclined to p_B, as shown in fig. 8;

then $\quad 2p_1 \cos\delta = p_B$ and $2p_1 \sin\delta = v_B\sqrt{\frac{K\rho}{g}}$ . . . (1)

Turning the vectors through the angle α in order to find the conditions at A,

$$2p_1 \cos(\alpha+\delta) = p_A, \text{ and } 2p_1 \sin(\alpha+\delta) = v_A\sqrt{\frac{K\rho}{g}}, \quad (2)$$

combining with (1)

$$\frac{\sin(\alpha+\delta)}{\sin\delta} = \frac{v_A}{v_B}$$

If $v_A = v_B$, $\sin(\alpha+\delta) = \sin\delta$,

or $(2n+1)\pi - (\alpha+\delta) = \delta$,

that is $\delta = \left(\frac{2n+1}{2}\right)\pi - \frac{\alpha}{2}$,

n being a whole number depending on the value of α.

$$\therefore \quad 2p_1 = \frac{v_A}{\sin\frac{1}{2}\{(2n+1)\pi+\alpha\}}\sqrt{\frac{K\rho}{g}},$$

and $$p_A = v_A\sqrt{\frac{K\rho}{g}}\cot\tfrac{1}{2}\{(2n+1)\pi+\alpha\},$$

$$p_B = v_A\sqrt{\frac{K\rho}{g}}\frac{\cos\frac{1}{2}\{(2n+1)\pi-\alpha\}}{\sin\frac{1}{2}\{(2n+1)\pi+\alpha\}}.$$

Thus if α is equal to some odd multiple of π, or in other words if l is equal to some odd multiple of half the wave-length, the pressures tend to become infinite.

(*c*) A sinoidal pressure-wave travelling along a pipe is accompanied by a sinoidal particle velocity wave in phase with it of which the maximum value is $p\sqrt{\frac{g}{K\rho}}$, where p is the maximum value of the pressure. Hence the power transmitted by such a wave is $\frac{p^2 s}{2}\sqrt{\frac{g}{K\rho}}$ if the sectional area of the pipe is s sq. inches.

In the case under consideration the motor is absorbing power at the rate of P inch-lb. per second.

Let the vectors $\overline{Op_1}$, $\overline{Op_2}$ (fig. 9) represent the forward and backward waves respectively as before. Let θ be the angle by which $\overline{Op_2}$ leads $\overline{Op_1}$ at B. Then, since the backward wave carries power back to the generator, the power absorbed

$$= P = \frac{(p_1^2 - p_2^2)}{2}s\sqrt{\frac{g}{K\rho}}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also $$v_B^2 = p_1^2 + p_2^2 - 2p_1p_2\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

On swinging the vectors backwards and forwards through the angle $\alpha\left(=\omega l\sqrt{\frac{\rho}{Kg}}\right)$ the equation

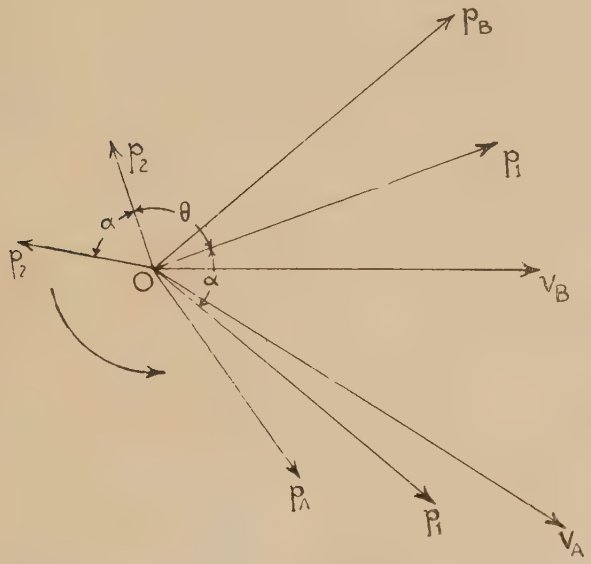
$$v_A^2 = p_1^2 + p_2^2 - 2p_1p_2\cos(\theta+2\alpha) \quad . \quad . \quad . \quad (3)$$

is obtained. Hence there are three equations by means of which the three unknowns p_1, p_2, and θ can be determined.

It is to be noticed that as p_1 and p_2 are unequal there is no tendency for the pressure to become infinite whatever the value of α.

With the addition of complications such as inertias or hydraulic capacities to the pipe line the investigation will naturally become more complex, but in every case equations equal in number to the unknowns can be obtained by the method set forth in this paper.

Fig. 9.



In the case where a pipe divides in order to supply two or more motors the line of investigation would be as follows. At each of the n motors there can be obtained two equations for the three unknowns p_1, p_2, and θ. If all the waves are transferred to the junction the composite pressures for each pipe at this point will be equal in magnitude and identical in phase; this fact will yield $(n-1)$ equations. The particle velocity in the generator pipe will be equal to the vectorial sum of the particle velocities in the n motor pipes. By transferring the waves to the generator, the particle velocity at which is known, another equation can be obtained and this equation will complete the $3n$ equations necessary to find the $3n$ unknowns.

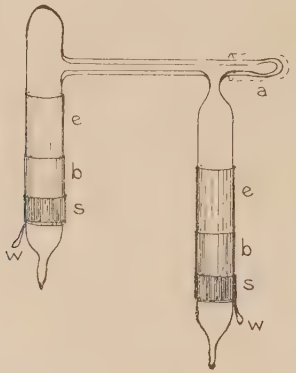
My grateful thanks are due to Professor G. B. Jeffery of King's College, London, for very valuable criticism.

XI. *Modified Vacuum Tubes.* *By* J. J. Manley, *M.A.*, *Research Fellow, Magdalen College, Oxford* *.

DURING the spectroscopic examination of small quantities of gas at very low pressures, two well-known defects of the vacuum tube were more than usually apparent. The two defects—(1) an insufficient luminosity for the feebler lines of spectra, and (2) contamination of the gas under examination by others emitted by the electrodes—were removed in the following way.

1. *Insufficient luminosity.*—From an end-on tube, light passes out in two equal portions which travel in opposite directions; hence 50 per cent. of the available light is, for spectroscopic work, ineffective. To avoid difficulties attending this great loss, the tube was altered and given the slightly

Fig. 1.



different form shown in fig. 1, and the limb *a* silvered to an extent represented by the enclosing dotted line. Under these circumstances, light which ordinarily emerges and becomes useless is reflected back, and so assists in illuminating the slit: thus the intensity is largely increased, and lines previously measured with difficulty are now so much enhanced in strength that their positions may be determined with ease. A further advantage is secured if, in addition to silvering the limb *a*, the capillary tube is treated as recommended by Merton and Johnson†. Maximum

* Communicated by the Author.

† Phil. Mag. xlvi. p. 448 (Sept. 1923).

intensity and uniformity of illumination are thus simultaneously realized.

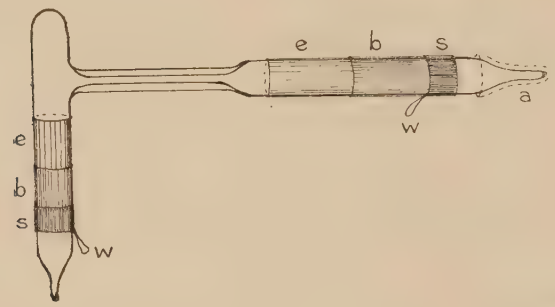
2. *Occluded Gases.*—The inconveniences arising from occluded gases were avoided by use of electrodes wholly external. This excellent plan, although not new, does not appear to be sufficiently well known or appreciated. Possibly this is because external electrodes as ordinarily attached are apt to lead to a piercing of the tube, and that within a comparatively short time. Numerous experiments have, however, led to the conclusion that by fulfilling two essential conditions, tubes having external electrodes are in every respect quite as reliable as others. The required conditions are (1) a certain massiveness for the electrodes, and (2) a complete absence of air-bubbles from the engaging surfaces of glass and electrode: these conditions may be secured in the following way.

First, the tube is chemically cleaned and dried both within and without. Next, that part of the discharge chamber upon which the electrode is to be built is thinly and evenly coated with concentrated and clear shellac varnish; then a strip of tin-foil 7 cm. wide and 20 cm. long is placed upon the varnish, with one of its ends parallel to the axis of the tube: the tube, held horizontally, is now slowly rotated so as to wind the foil; as the winding proceeds, the foil is gently and continuously rubbed in order to expel the enclosed air. When the foil covers some three-quarters of the circumference of the tube, the inner surface of the unwound portion is coated with varnish, and the winding and rubbing resumed and continued until all the foil has been attached to the tube: finally, the electrode *e* thus built up is varnished and, when dry, given a four-fold band *b* of tin-foil. At one end of this band, which is 3 or 4 cm. wide, is embedded a looped copper wire *w*, which serves as a connector for an induction coil. The band and its wire are tightly strapped to the electrode by two turns of adhesive tape *s*, and a last coat of varnish applied. An end-on vacuum tube silvered at *a* and fitted with external electrodes is represented in fig. 1. The tube is of "Duro" glass; its discharge chambers are 9 cm. long and 1·5 cm. wide. Tubes of this kind have for months been frequently used for prolonged periods; and although the coil operating the tubes yields a powerful 1-inch spark, in no instance has a tube broken down or become in any way visibly defective. Such tubes are made with ease, and the use of expensive platinum wire avoided.

3. *Some further simplifications.*—Two other modifications

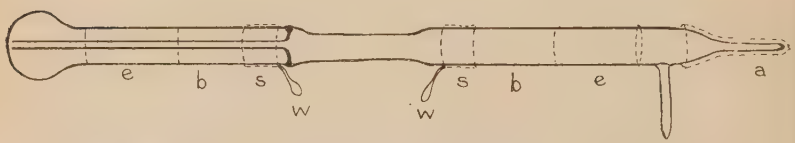
of the end-on tube are shown in figs. 2 and 3. That seen in fig. 2 involves the making of one T-joint only. It is more

Fig. 2.



efficient than that shown in fig. 1. The increased efficiency is due to the reflecting power of the inner surface of the tin-foil forming the external electrode of the horizontal chamber. Here, in a secondary degree, the tin-foil functions in much the same way as does the silvered surface of Merton and Johnson's tube (*vide supra*).

Fig. 3.



The efficiency of the tube represented in fig. 3 is superior to that of the other two: the reason is threefold. (1) In this form the *whole* glow is viewed end-on; (2) *both* electrodes now function after the manner of Merton and Johnson's silvered surface; and (3) the end of the capillary tube viewed by the spectroscope can be brought very close to the slit. In connexion with this last point, we observe that distortion must be absent from that portion of the bulb confronting the capillary tube. The desired uniformity may be obtained by re-forming the bulb some four or five times. The terminal limbs *a* of tubes 2 and 3 are, like that of tube 1, silvered to an extent indicated by the enveloping dotted lines. The letters appended to figs. 2 and 3 retain the meanings assigned in fig. 1.

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XII. *Monoplane Theory.*

By N. K. Bose, *Ph.D.*(*Göttingen*) *.

(1)

LET us suppose that there is an infinite volume of frictionless and incompressible fluid moving in an infinite space with a constant velocity V and that in this fluid we place a plane, the length of the plane being at right angles to the direction of the free fluid. Now what happens has been described by Prof. Prandtl in his paper "Tragflügeltheorie." It is known from simple experimental facts that to keep the aeroplane fixed in this moving fluid a certain force must be applied. Now whence comes this force which must have been applied by the fluid on the body as we have supposed the fluid to be frictionless and incompressible? To account for this fact Kutta assumed that there is a circulation of liquid round the aeroplane. From our conceptions of classical Hydrodynamics we know that circulation round a closed fluid line cannot change nor can it be generated by any known forces. The explanation of why, in spite of this, the aeroplane gained circulation is given by Prof. Prandtl as follows: At the first moment of the motion there is still no circulation, there is only a flow at high velocity around the trailing edge. This motion cannot however continue: there is instantly formed at the trailing edge a vortex of increasing intensity, which in accordance with a theorem of Helmholtz that the vortex is made up of the same fluid particles remains with the fluid as it passes on. The circulation round the wing and the vortex taken together remains equal to zero, there remains then round the wing a circulation equal and opposite to that of the vortex which has gone off with the current.

To give this idea a mathematical basis, what is necessary is to divide the aeroplane surface into so many "differential surfaces" so that each small elemental surface may be looked upon as an aeroplane on which fluid pressure is calculated independently of all other elements, and then it is integrated throughout the whole volume occupied by these elemental aeroplanes. The conception is something like this: at each elemental aeroplane there is a circulation which is denoted by γ, and from the edges of this element vortices flow off with the fluid; the strength of the elemental vortices is ϵ. So we can picture the thing like this: under each differential aeroplane there is a vortex of strength γ which is fixed with

* Communicated by Prof. C. Runge, the University of Göttingen.

the element and from each element a vortex line is flowing off with the liquid whose strength per unit volume is ϵ, so that the aeroplane becomes the seat of innumerable fixed vortices of strength γ per unit volume, and from the aeroplane free vortices are flowing off with the liquid like ribbons with strength ϵ per unit volume.

With this conception Prof. Prandtl finds the force on each differential surface as $\rho(\gamma \times \mathfrak{V})$ per unit volume, where ρ is the density of the fluid, γ is the vortex strength at that element, and $\mathfrak{V}$ is the velocity of the liquid at that element, so that the force with which the aeroplane must be held to keep it fixed is given by

$$K = \rho \iiint (\gamma \times \mathfrak{V})\, dx\, dy\, dz,$$

where the integration is throughout the whole volume filled by the vortices.

Now let us suppose that the bound vortices give rise to a velocity V_1 in the liquid, and the free vortices ϵ to V_2, so that

$$\mathfrak{V} = V + V_1 + V_2 = V + V^*,$$

so that V^* is the velocity superimposed on the free fluid due to the presence of the aeroplane with its vortices. Since $\gamma + \epsilon$ gives rise to the velocity V^*, then

$$\gamma + \epsilon = \text{curl } V^* = \nabla \times V^* ;$$

also we know that V is constant, so

$$\nabla \times V = 0 ;$$

and since the free vortices are flowing off with the liquid so that the axis of ϵ is always parallel to $\mathfrak{V}$,

$$\epsilon \times \mathfrak{V} = 0 \quad \text{or} \quad (\nabla \times V_2) \times \mathfrak{V} = 0,$$

so that

$$\begin{aligned} K &= \rho \iiint (\gamma \times \mathfrak{V})\, dx\, dy\, dz \\ &= \rho \iiint (\nabla \times V_1) \times (V + V^*)\, dx\, dy\, dz \\ &= \rho \iiint (\nabla \times V^*) \times (V + V^*)\, dx\, dy\, dz \\ &\qquad \text{since } (\nabla \times V_2) \times \mathfrak{V} = 0, \\ &= \rho \iiint (\nabla \times V^*) \times V\, dx\, dy\, dz + \rho \iiint (\nabla \times V^*) \\ &\qquad \times V^*\, dx\, dy\, dz. \end{aligned}$$

Now the first integral can be written

$$\rho\left\{\iiint(\nabla\times \mathrm{V}^*)\,dx\,dy\,dz\right\}\times \mathrm{V} = \rho(\Gamma+\mathrm{E})\times \mathrm{V}^*,$$

where Γ and E are single equivalent vortices that can replace the bound and the free vortices respectively for all purposes.

For the second integral

$$\mathrm{I} = \iiint(\nabla\times \mathrm{V}^*)\times \mathrm{V}^*\,dx\,dy\,dz$$

we have from properties of Vector analysis

$$\begin{aligned}(\nabla\times \mathrm{V}^*)\times \mathrm{V}^* &= -\mathrm{V}^*\times(\nabla\times \mathrm{V}^*)\\ &= -(\mathrm{V}^*\,.\,\mathrm{V}^*)\nabla + (\mathrm{V}^*\,.\,\nabla)\mathrm{V}^*\\ &= (\mathrm{V}^*\,.\,\nabla)\mathrm{V}^* - \tfrac{1}{2}\nabla(\mathrm{V}^*\,.\,\mathrm{V}^*).\end{aligned}$$

If u^*, v^*, w^* be the components of V* along the three coordinate axes and i, j, k be three unit vectors along these axes, then

$$\begin{aligned}(\nabla\times \mathrm{V}^*)\times \mathrm{V}^* = &\left(u^*\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} + w^*\frac{\partial u^*}{\partial z}\right)i\\ &+\left(u^*\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} + w^*\frac{\partial v^*}{\partial z}\right)j\\ &+\left(u^*\frac{\partial w}{\partial x} + v^*\frac{\partial w^*}{\partial y} + w^*\frac{\partial w^*}{\partial z}\right)k - \tfrac{1}{2}\nabla(q^{*2}),\end{aligned}$$

where q^* is the scalar value of V*.

Hence

$$\begin{aligned}\mathrm{I} = &\, i\iiint\left(u^*\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y} + w^*\frac{\partial u^*}{\partial z}\right)dx\,dy\,dz\\ &+ j\iiint\left(u^*\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y} + w^*\frac{\partial v^*}{\partial z}\right)dx\,dy\,dz\\ &+ k\iiint\left(u^*\frac{\partial w^*}{\partial x} + v^*\frac{\partial w^*}{\partial y} + w^*\frac{\partial w^*}{\partial z}\right)dx\,dy\,dz\\ &-\tfrac{1}{2}\iiint\nabla(q^{*2})\,dx\,dy\,dz.\end{aligned}$$

Now to integrate

$$\iiint \left(u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} + w^* \frac{\partial u^*}{\partial z}\right) dx\, dy\, dz$$

$$= \iiint u^* \frac{\partial u^*}{\partial x}\, dx\, dy\, dz + \iiint v^* \frac{\partial u^*}{\partial y}\, dy\, dz\, dx + \iiint w^* \frac{\partial u^*}{\partial z}\, dz\, dx\, dy.$$

Integrating by parts

$$= \iint \left\{ u^* u^* - \int u^* \frac{\partial u^*}{\partial x}\, dx \right\} dy\, dz + \iint \left\{ u^* v^* - \int u^* \frac{\partial v^*}{\partial y}\, dy \right\} dz\, dx + \iint \left\{ w^* u^* - \int u^* \frac{\partial w}{\partial z}\, dz \right\} dx\, dy.$$

Now if we conceive of a surface $d\mathrm{S}$ at (x, y, z), if l, m, n be the direction cosines of its normal,

$$dy\, dz = l\, d\mathrm{S}, \quad dz\, dx = m\, d\mathrm{S}, \quad dx\, dy = n\, d\mathrm{S},$$

so that

$$\iiint \left(u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} + w^* \frac{\partial u^*}{\partial z}\right) dx\, dy\, dz = \iint (lu^* + mv^* + nw^*)\, u^*\, d\mathrm{S} - \iiint \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\right) u^*\, dz\, dy\, dz.$$

Now

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} = 0,$$

as the liquid is homogeneous and incompressible.

Hence

$$\mathrm{I} = \iint (lu^* + mv^* + nw^*)\, (u^* i + v^* j + w^* k)\, d\mathrm{S} - \frac{1}{2} \iiint \nabla (q^{*2})\, dx\, dy\, dz$$

$$= \iint (lu^* + mv^* + nw^*)\, \mathrm{V}^*\, d\mathrm{S} - \frac{1}{2} \iiint \nabla (q^{*2})\, dx\, dy\, dz,$$

where the surface integral is over a surface enclosing all the vortices and the volume integral throughout a space covered by the vortices.

Now $$\nabla(q^{*2}) = 2q^* \nabla(q^*),$$

$$\therefore \iiint \nabla(q^{*2})\, dx\, dy\, dz = 2 \iiint q^* \nabla(q^*)\, dx\, dy\, dz$$

$$= 2i \iiint q^* \frac{\partial q^*}{\partial x}\, dx\, dy\, dz + 2j \iiint q^* \frac{\partial q^*}{\partial y}\, dy\, dz\, dx + 2k \iiint q^* \frac{\partial q^*}{\partial z}\, dz\, dx\, dy$$

$$= i \iint l q^{*2}\, dS + j \iint m\, q^{*2}\, dS + k \iint n\, q^{*2}\, dS,$$

$$= \iint (li + mj + nk)\, q^{*2}\, dS = \iint N\, q^{*2}\, dS,$$

where N is the unit normal vector to the element of surface. So that

$$I = \iint (lu^* + mv^* + nw^*)\, V^*\, dS - \frac{1}{2} \iint N\, q^{*2}\, dS,$$

then

$$K = \rho(\Gamma + E) \times V + \rho \iint (lu^* + mv^* + nw^*)\, V^*\, dS - \tfrac{1}{2}\rho \iint N\, q^{*2}\, dS.$$

Now Γ is Γl of the Kutta-Joukowski formula for lift, so that $\rho(\Gamma \times V)$ is a force perpendicular to V, and according to Prof. Gibbs's conception if V be horizontal, from left to right, and Γ along the length of the aeroplane from behind forward, then the force is vertically downward, hence it is opposite to "lift," and equivalent in amount to it. The force $\rho(E \times V)$ is also perpendicular to V, hence its component along V is zero, and it being of a higher order of small quantities than $\rho(\Gamma \times V)$, its effect on the latter force, *i. e.* the lift, is negligible. Hence to a first order of approximation

$$E \times V = 0.$$

Now the integral

$$\rho \iiint (\gamma \times \mathfrak{B})\, dx\, dy\, dz$$

is to be integrated throughout the whole space covered by the vortices. But if we extend the limits of integration so as

to make them infinite we do not thereby change the value of the integral, because outside the space where the vortices are the value of the integrand is zero as at every point there $\gamma = 0$. So we can take all the integrals over infinite space. For the surface integrals let us assume that we enclose the volume by a cylindrical surface and two parallel planes so that the axis of the cylinder is parallel to V and passes through the aeroplane and the two planes ; one, A_1, is in front of the aeroplane and the other, A_2, is behind the aeroplane and they are perpendicular to the axis of the cylinder. If now we take the radius R of the cylinder very great, both the surface integrals would vanish over it since the velocities are of order $\frac{1}{R^2}$ and dS of order R^2. Also the values of these two surface integrals over A_1 will vanish if we take A_1 sufficiently far off from the aeroplane. As to the values of these integrals over A_2, we must remark that the velocities q^* V* u^* v^* w^* are composed of two parts ; one is due to the bound vortices and the other one to the free vortices, so that

$$q^* = q_1 + q_2, \quad V^* = V_1 + V_2, \quad u^* = u_1 + u_2,$$
$$v^* = v_1 + v_2, \quad w^* = w_1 + w_2.$$

Now we see that as we move the plane A_2 further and further away from the aeroplane always parallel to itself, the quantities q^* V* u^* v^* w^* approximate more and more to q_2 V_2 u_2 v_2 w_2 and the influence of the bound vortices becomes feebler and feebler, so that ultimately when the plane A_2 is far off enough from the aeroplane then we can take with sufficient degree of approximation

$$q^* = q_2, \quad V^* = V_2, \quad u^* = u_2, \quad v^* = v_2, \quad w^* = w_2,$$

and the surface integrals become

$$\rho \iint (lu_2 + mv_2 + nw_2) V_2 dS - \tfrac{1}{2} \rho N \iint q_2^2 \, dS.$$

Now $lu_2 + mv_2 + nw_2$ is the component of the velocity due to the free vortices normal to A_2, but we have so taken the plane A_2 that it is perpendicular to V, *i. e.* perpendicular to the direction of the vortices which are parallel to each other and to V when far off from the aeroplane, so that the normal components of the velocities generated by these free vortices are zero, *i. e.* the first of the above two integrals are zero.

Hence we get

$$K = \rho(\Gamma \times V) - \tfrac{1}{2} \rho N \iint q_2^2 \, dS.$$

Thus we see that to keep the aeroplane in position this force K must be applied on the aeroplane, which easily decomposes itself into two parts perpendicular to each other. Thus the air force on the aeroplane can be decomposed into two parts: one is what is called Lift and is equal to $\rho(\Gamma \times V)$ and is perpendicular to V and upwards; whereas the other portion is what is called Drag and is equal to $\frac{1}{2}\rho \iint q_2^2 \, dS$ and is parallel and in the same direction as V. From this expression of drag it is at once apparent that drag is exactly equivalent to that portion of the energy which is flowing off with the liquid in the free vortices and will be ultimately dissipated into the air as heat. That the energy is exactly equivalent to the drag found by Prof. Prandtl will be shown afterwards. It may be of interest to note that the integral (11) in the "Tragflügel-theorie," which Prof. Prandtl has assumed to be zero only when the bound vortices are parallel to each other, can be shown to be always zero by means of the foregoing vector identity

$$(\nabla \times V) \times V = (V \cdot \nabla)V - \tfrac{1}{2}\nabla(V \cdot V),$$

and the (11) integral can be reduced to

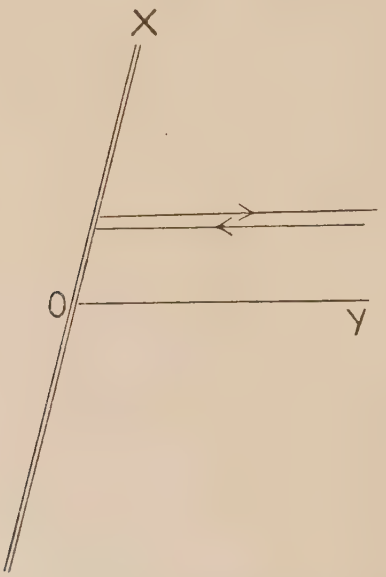
$$\iiint (\nabla \times V_1) \times V_1 \, dx \, dy \, dz. \tag{2}$$

To find the velocity-potential and other characteristics of the vortex system that is flowing off with the liquid we shall have to refer again to another conception of Prof. Prandtl. He found that the density of the lift (lift per unit length) is not constant over the whole space of the aeroplane, but in general falls off gradually from a maximum at the middle to zero at the ends. In accordance with what has been proved there corresponds to this a circulation decreasing from within outwards. Therefore, according to the theorem that by the displacement of the closed curve the circulation Γ can only change if a corresponding quantity of vortex filaments are cut, we must assume that vortex filaments proceed off from the trailing edge wherever Γ changes. For a portion of this edge of length dx, the vortex strength that is flowing off is therefore to be written $\frac{d\Gamma}{dx} dx$, hence per unit length of the edge the strength of the free vortices is $\frac{d\Gamma}{dx}$. These vortex filaments flowing off, closely side by side, form

taken as a whole a surface-like figure which is known as "Vortex Sheet."

The strengths of our vortex sheet remain unchanged during the whole flight, yet the separate parts of the sheet influence each other and there takes place a gradual rolling up of the sheet. An exact theoretical investigation of this phenomenon has not been made yet; it can only be said that the two halves of the sheet become concentrated more and more, and finally at great distances from the wing there remain a pair of vortices with rather weak core. As a first approximation we shall not consider this latter development of the sheet, but shall proceed with the calculations as if all the vortex filaments were running off behind in straight lines opposite to the direction of flight.

Fig. 1.



Now to treat the subject mathematically we consider the vortex sheet as a plane, say x–y plane, and take the axis of y parallel to the vortex filaments and the x-axis along the length of the aeroplane, and take the origin of coordinates O at the middle point of the aeroplane, so that on the positive side of the x-axis all the free vortices have a clockwise rotation when viewed along this direction of flight, and on the negative side of the x-axis a counter-clockwise rotation. Let

us take a vortex filament ABC of strength $\frac{d\Gamma}{dx}dx$ at A$(x, 0, 0)$ and find out ϕ, ψ, u, v, w due to it at a point P$(x'\ y'\ z')$ in the free liquid.

We know from the properties of a single straight vortex, on one side going to infinity and on the other side ending on a surface,

$$\psi = \frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}dx\log r$$

and

$$\phi = -\frac{1+\cos\alpha}{4\pi}\cdot\frac{d\Gamma}{dx}dx\,.\,\theta,$$

where θ increases with the circulation due to the vortex filament and r the distance of P from the filament, and $\alpha = \angle$ ABP.

$$\therefore \quad w = \phi + i\psi = i\frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}\log re^{i\theta}\,.\,dx$$

$$= i\frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}\log(r\cos\theta + ir\sin\theta)\,dx$$

$$= i\frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}\log(x-x'+iz')\,dx$$

$$= i\frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}\log(x-\overline{x'-iz'})\,dx$$

$$= i\frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}\log(x-\mathrm{R}e^{-i\Theta})\,dx.$$

$$\therefore \quad \psi = \text{real part of } \frac{1+\cos\alpha}{4\pi}\frac{d\Gamma}{dx}\log(x-\mathrm{R}e^{-i\Theta})\,dx.$$

where R and Θ are quantities corresponding to r and θ with respect to the y-axis.

In the case when P$(x'\ y'\ z')$ is very far off from the aeroplane so that $\cos\alpha = 1$, Ψ ultimately becomes

$$\Psi = \int_{-\frac{b}{2}}^{\frac{b}{2}} \text{real part of } \frac{1}{2\pi}\frac{d\Gamma}{dx}\log(x-\mathrm{R}e^{-i\Theta})\,dx.$$

Integrating by parts,

$$\Psi = \text{real part of } \frac{1}{2\pi}\left[\Gamma \log (x - \mathrm{R}e^{-i\Theta})\right]_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$- \text{real part of } \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma dx}{x - \mathrm{R}e^{-i\Theta}},$$

since Γ is zero at both the limits

$$\Psi = \text{real part of } \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma dx}{-x + \mathrm{R}e^{-i\Theta}}$$

$$= \text{real part of } \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma dx}{(x' - x) - iz'}$$

$$= \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma(x' - x)\, dx}{x^2 + \mathrm{R}^2 - 2xx'}$$

$$= -\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma \cos\theta}{r}\, dx.$$

Similarly

$$\Phi = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma z'\, dx}{x^2 + \mathrm{R}^2 - 2xx'}$$

$$= \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma \sin\theta}{r}\, dx.$$

In this case to find the value of u, v, w we see that since the vortices are all parallel to the y-axis,

$$u \text{ (due to one vortex filament)} = -\frac{\partial \psi}{\partial z}$$

$$= -\text{real part of } \frac{\partial}{\partial z} \log (x - \mathrm{R}e^{-i\theta}) \frac{d\Gamma}{dx} \frac{dx}{2\pi}$$

$$= -\text{real part of } \frac{i}{x - \mathrm{R}e^{-i\theta}} \frac{d\Gamma}{dx} \frac{dx}{2\pi}$$

$$= -\frac{1}{2\pi} \cdot \frac{z'}{x^2 + \mathrm{R}^2 - 2xx'} \frac{d\Gamma}{dx}\, dx.$$

$$\therefore \quad U = \Sigma u = -\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{z'}{x^2+R^2-2xx'}\frac{d\Gamma}{dx}\,dx$$

$$= -\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{2z'(x-x')\Gamma}{(x^2+R^2-2xx')^2}dx,$$

since Γ is zero at the limits

$$= -\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\sin 2\theta}{r^2}\Gamma\,dx.$$

Similarly,

$$W = \Sigma w = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\cos 2\theta}{r^2}\Gamma\,dx.$$

Hence, when we go far off from the aeroplane but not far off from the vortex sheet we get as the characteristics of the vortex sheet

$$\Phi = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\sin\theta}{r}\Gamma\,dx, \qquad \Phi = -\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\cos\theta}{r}\Gamma\,dx,$$

$$U = -\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\sin 2\theta}{r^2}\Gamma\,dx, \quad W = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\cos 2\theta}{r^2}\Gamma\,dx,$$

and $v = 0$,

where "r" is the distance of the point P from the vortex filament $\frac{d\Gamma}{dx}dx$, and θ the angle which r makes with the negative direction of the x-axis.

We can get the corresponding quantities on the vertical plane through the aeroplane only if we halve these values of ϕ, ψ, u, v, w, as the vortex sheet is at a point of the plane only on one side.

3.

We have found before that the resistance of an aeroplane to the liquid motion can be completely represented by the amount of energy that is flowing off with the fluid on account of the free vortices going along with it. We can now show that this energy is exactly equivalent to the expression which Prof. Prandtl has found in his "Tragflügeltheorie"; thus we shall prove conclusively that it is the energy that is flowing off with the free vortices that constitutes the resistance of the aeroplane.

We have

$$\int q^2\,dS = \iint \left\{ \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right\} dx\,dz\,;$$

as we have seen before, dS is an element of a plane which is parallel to the xz-plane and very far away from the aeroplane so that we can use the second values of ϕ, ψ, u, v, w, etc.

$$\therefore \quad \int q^2\,dS = -\int u \frac{\partial \phi}{\partial x}\,dx\,dz - \iint w \frac{\partial \phi}{\partial z}\,dz\,dx.$$

Integrating by parts

$$= -\int u\phi\,dz + \iint \phi \frac{\partial u}{\partial z}\,dx\,dz - \int w\phi\,dx + \iint \phi \frac{\partial w}{\partial z}\,dz\,dx$$

$$= -\int (lu + nw)\phi\,ds + \iint \phi \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) dx\,dz,$$

where l and n are the direction cosines of the normal of the element of length ds on the plane; and since the fluid is incompressible and homogeneous

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

$$\therefore \quad \int q^2\,dS = -\int (lu + nw)\phi\,ds$$

taken over a line enclosing the free vortex sheet; but as there is a discontinuity in the fluid we must take this line-integral over two close lines, one just enclosing the vortex sheet so as to exclude it from the field of consideration and another enclosing the whole system; both these lines are in the plane. If now we extend the latter line so that it becomes an infinitely great circle the value of the integral over it is zero, as when R is very great u and w are of $1/R^2$ order, ϕ of $1/R$, and ds of R, so that ultimately we are left with the line just enclosing the vortex sheet only.

Let us take the line as the perimeter of a rectangle just enclosing the sheet so that its length is along the breadth of the sheet and of length b and its breadth along the thickness of the sheet and of length dz; at any point on the length of the rectangle l is zero and n is 1 for the upper line and -1 for the lower line; and at any point on the breadth of the rectangle n is zero and $u=0$, so that the values of the integrand are zero at these two short vertical lines. Hence we are left with the value of the integral over the two horizontal parallel lines.

$$\therefore \quad \int q^2\,dS = \int (w_1\phi_1 - w_2\phi_2)\,ds,$$

where $w_1\phi_1$ and $w_2\phi_2$ are the values of w and ϕ at the lower line and at a corresponding point on the upper line. But from the above equations for ϕ, ψ, u, w we see that w is continuous through the vortex sheet, *i. e.* if we take two corresponding points on opposite sides of the sheet very close to the sheet, then $w_1=w_2$. Hence

$$\int q^2\,dS = \int w(\phi_1-\phi_2)\,dx\ ;$$

we can now take ds as equivalent to dx.

But we find from the above value of ϕ that it is discontinuous, *i. e.*, it has quite opposite values on two corresponding points on opposite sides of the vortex sheet; we have as before

$$\phi = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\sin\theta}{r}\,\Gamma\,dx.$$

We see that $dx\sin\theta$ is the projection of dx on a line perpendicular to r, i. e. $\dfrac{dx\sin\theta}{r}$ is the small angle subtended at P by dx; if we denote this angle by $d\alpha$, then at any point (P) due to the vortex filament at x

$$\phi = \frac{1}{2\pi}\,\Gamma\,d\alpha.$$

If now we take P just below the filament on the rectangle, and then just above the filament on the rectangle, then the difference of the angle $d\alpha$ at the two points is 2π,

$$i.\,e.,\quad \phi_1-\phi_2 = \Gamma.$$

Therefore the energy flowing off with the liquid as free vortices is

$$\tfrac{1}{2}\rho\int q^2\,dS = \tfrac{1}{2}\rho\int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma w\,dx$$

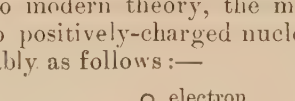
$$= \frac{1}{4\pi}\rho\iint_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\cos 2\theta}{r^2}\,\Gamma_1\Gamma_2\,dx_1dx_2.$$

This is exactly the expression found by Prof. Prandtl in the second part of his communication "Tragflügeltheorie." The above expression can never become indeterminate as r is the distance of a point $(x_1\ 0)$ on the vortex at x_1, from a point $(x_2\ z_2)$ on the line which is parallel to the x-axis but at a distance z_2 below in the same vertical plane so that it is just outside the vortex sheet; so that $r^2=(x_1-x_2)^2+z_2^2$ and can never become zero however small z_2 may be.

XIII. *Atomic Dimensions and Gaseous Hydride Formation.* *By* H. L. Riley, *B.Sc., D.I.C., Beit Scientific Research Fellow, Demonstrator in Chemistry at the Imperial College of Science and Technology* *.

THE tendency of two elements to react chemically is dependent on several factors, the chief of which are the orientation of the electrons and the relative dimensions of the atoms or molecules concerned in the reaction. The formation of a gaseous hydride is probably the simplest form of chemical reaction, especially when one considers the formation of the hydride of a gaseous element of low atomic weight. The study of such reactions is most likely to give us some insight into the mechanism of chemical reactivity. The following is an attempt to connect the tendency to hydride formation with atomic dimensions.

According to modern theory, the molecule of hydrogen consists of two positively-charged nuclei and two electrons, arranged probably as follows :—



The two electrons are thought to revolve in some orbit between the two nuclei. Such an arrangement suggests that the molecule of hydrogen will act as a sphere, or as something approximating to a sphere. Hence it is possible, from the known volume of the molecule, to obtain an approximate value of its diameter. This works out to be $2{\cdot}5 \times 10^{-8}$. (Calculated from the mean value of the volume given by the "b" in van der Waal's equation, the viscosity, the thermal conductivity, and the mean free path.)

If one assumes that this value is an approximate measure of the diameter of the electron orbit in the hydrogen molecule, and compares it with the atomic diameters of those elements which form gaseous hydrides, the following interesting fact becomes evident. All those elements which have an atomic diameter less than $2{\cdot}5 \times 10^{-8}$ form hydrides with the evolution of energy ; and those with an atomic diameter greater than the above value form hydrides with the absorption of energy. The atomic diameters—*i. e.*, the diameter of the outer shell of electrons—of those elements which form gaseous hydrides, together with the heats of formation of their respective hydrides, are appended.

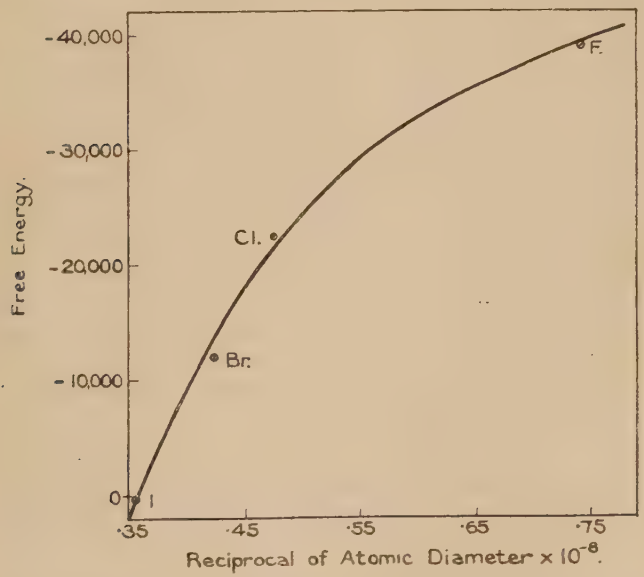
It is not to be expected that there will be any simple quantitative relationship between the atomic diameter and

* Communicated by Prof. H. B. Baker, C.B.E., M.A., D.Sc., F.R.S.

the heat of formation of the hydride, as the latter is dependent on such factors as temperature and molecular aggregation, being only an approximate measure of the tendency to chemical reaction.

The above seems to indicate that the tendency of an element to form a gaseous hydride is dependent chiefly on the diameter of the atom. If this is less than the diameter of the electron orbit in the hydrogen molecule, then this tendency will be great, because the reacting atom is of such a size that it can easily pierce the orbit and in so doing upset the stability of the electron arrangement in the reacting molecules. The reverse will, of course, be the case when the diameter of the reacting atom is greater than the diameter of the electron orbit in the hydrogen molecule.

Fig. 1.



It may also be pointed out that all elements with an atomic diameter of greater than $2{\cdot}5 \times 10^{-8}$ either absorb energy during formation of the hydride or do not react with hydrogen to form a gaseous hydride.

That there probably is some simple relationship between atomic dimensions and the tendency to gaseous hydride formation can be seen in the case of fluorine, chlorine, bromine, and iodine by plotting the reciprocal of the atomic diameter of the halogen against the free energy of the halogen acid.

It is also interesting to compare the heats of formation of methane, ammonia, water, and hydrofluoric acid.

Hydride	CH_4.	NH_3.	$HO_{\frac{1}{2}}$.	HF.
Heat of formation	5·4	4·0	29·0	38·6
Atomic diameter $\times 10^8$	1·54	1·59	1·49	1·35

It will be seen that the heats of formation are in the same order as the reciprocals of the atomic diameters, and this in spite of the fact that the heat of formation of methane is from solid amorphous carbon.

If the above is coincidence, being the relationship between two factors, heat of formation and atomic diameter, both periodic functions of the atomic number, then it is exceedingly difficult to picture the mechanism of a chemical reaction of the above type. The hydrogen molecule is an extremely stable structure, dissociation only occurring partially at the highest attainable temperatures. That another atom is capable of upsetting such a stable arrangement can only be understood when one pictures the atom actually entering the molecule of hydrogen, when electron rearrangement takes place. Energy will be evolved, if the energy liberated by the electron rearrangement is greater than that required to increase the diameter of the electron orbit in the hydrogen molecule, to permit the entry of the reacting atom.

Element.	At. Diam. $\times 10^8$*.	Heat of Formation.	
Carbon	1·54	$C_{(amorph.)} + 2H_2 = CH_4$,	+21·7
Nitrogen	1·59	$\frac{1}{2}N_2 + \frac{3}{2}H_2 = NH_3$,	+12·0
Oxygen	1·49	$\frac{1}{2}O_2 + H_2 = H_2O$,	+58·1
Fluorine	1·35	$\frac{1}{2}F_2 + \frac{1}{2}H_2 = HF$,	+38·6
Silicon	2·35	$Si_{(amorph.)} + 2H_2 = SiH_4$,	+ 8·7
Phosphorus	2·0	$P_{(white)} + \frac{3}{2}H_2 = PH_3$,	+ 4·9
Sulphur.....	2·05	$S_{(rhombic)} + H_2 = H_2S$,	+ 2·7
Chlorine	2·10	$\frac{1}{2}Cl_2 + \frac{1}{2}H_2 = HCl$,	+22·0
Arsenic	2·52	$As_{(cryst.)} + \frac{3}{2}H_2 = AsH_3$,	−44·2
Selenium	2·35	$Se_{(amorph.)} + H_2 = SeH_2$,	+ 6·0
Bromine	2·38	$\frac{1}{2}Br_2 + \frac{1}{2}H_2 = HBr$,	+ 8·4
Antimony	2·80	$Sb_{(metal)} + \frac{3}{2}H_2 = SbH_3$,	−87
Tellurium	2·65	$Te_{(metal)} + H_2 = H_2Te$,	−34·9
Iodine	2·80	$\frac{1}{2}I_2 + \frac{1}{2}H_2 = HI$,	− 6·1
Bismuth	2·96	.....	

* Prof. W. L. Bragg, Phil. Mag. (6) xl. p. 169 (1920).

XIV. *Coupled Cord Pendulums.*

By Charles H. Lees, *D.Sc.*, *F.R.S.**

1. THE interest which has been aroused by oscillograph investigations of the electrical oscillations in the coupled circuits used in wireless telegraphy has led to a revival of the study of coupled dynamical systems in general, and new systems have been invented and their properties studied, either theoretically or experimentally, in order to facilitate the subsequent study of the properties of coupled electrical circuits †.

These studies have added much to our knowledge of the consequences of dynamical coupling, but the results of the theoretical work are not always given in a form which admits of the principal features of the oscillations being readily recognized, nor is it in general easy to see why the experimental curves reproduced have their particular forms under the conditions stated.

The following pages attempt to present the theory of one of the simplest forms of coupled pendulums—that used by Barton and Browning in most of their work,—and to show that by neglecting the small damping which occurs, by using a symmetrical notation and dealing in the fundamental portion of the theory with inclinations rather than with displacements, the work may be reduced to the determination of a certain length which enters into the expression of the periods and amplitudes of the component oscillations which make up the complete oscillation of each pendulum, and that the solution of the problem, so far as it is independent of the initial conditions of the oscillations, can be obtained graphically.

2. Let two simple pendulums (fig. 1) of lengths h_1 h_2 where $h_2 > h_1$ be suspended from two points S_1 S_2 in the same horizontal plane, and let two points B_1 B_2 on the two pendulums in a horizontal plane h below the plane of support be joined by an inextensible rod of length equal to S_1S_2 and of mass negligible in comparison with m_1 and m_2, the masses of the bobs of the pendulums. Let the oscillations take place in the vertical plane through S_1S_2, then the figure $S_1S_2B_2B_1$ remains a parallelogram throughout the motion.

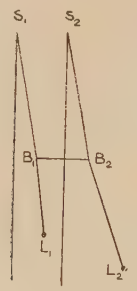
* Communicated by the Author.

† See *e.g.* Lyle, Phil. Mag. xxv. p. 567 (1913); Barton & Browning, Phil. Mag. xxxiv. p. 246 (1917), xxxv. p. 62 (1918), xxxvi. p. 36 (1918); Plummer, Phil. Mag. xxxiv. p. 510 (1917); Jackson, Phil. Mag. xxxix. p. 284 and xl. p. 329 (1920); Narayan, Phil. Mag. xliii. p. 567 (1922).

Let $\theta\,\theta_1\,\theta_2$ be the small inclinations to the vertical of the common portion of each pendulum and the independent portions respectively at any time, then the displacements $x_1\,x_2$ of the bobs of the pendulums are

$$\left.\begin{aligned} x_1 &= h\theta + (h_1 - h)\theta_1 \\ \text{and} \qquad x_2 &= h\theta + (h_2 - h)\theta_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

Fig. 1.



The coupled pendulums $h_1 = S_1L_1 = 13$, $h_2 = S_2L_2 = 15$, $h = S_1B_1 = S_2B_2 = 8$, mass of L_1 : mass of $L_2 = 3 : 5$.

For equilibrium at the points B we must have, since the inclinations to the vertical are small,

$$\frac{m_1 + m_2}{\theta_2 - \theta_1} = \frac{m_1}{\theta_2 - \theta} = \frac{m_2}{\theta - \theta_1}.$$

Hence

$$\theta = \frac{m_1}{m_1 + m_2}\theta_1 + \frac{m_2}{m_1 + m_2}\theta_2$$

and

$$\left.\begin{aligned} x_1 &= (h_1 - b_2)\theta_1 + b_2\theta_2 \\ x_2 &= (h_2 - b_1)\theta_2 + b_1\theta_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

where $b_1 = \dfrac{m_1}{m_1 + m_2}h$ and $b_2 = \dfrac{m_2}{m_1 + m_2}h$, may each be looked on as the fraction of h the common length, prevented by each pendulum from being directly effective as a portion of the length of the other pendulum

Hence

$$\left.\begin{aligned} \ddot{x}_1 &= (h_1-b_2)\ddot{\theta}_1+b_2\ddot{\theta}_2 \\ \text{and}\quad \ddot{x}_2 &= (h_2-b_1)\ddot{\theta}_2+b_1\ddot{\theta}_1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

where $\ddot{x}$ signifies differentiation of x twice with respect to t.

The equations of motion of the centres of oscillation of the two pendulums are respectively

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -m_1 g\theta_1 \\ \text{and}\quad m_2\ddot{x}_2 &= -m_2 g\theta_2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

which become on substitution for $\ddot{x}_1$ and $\ddot{x}_2$

$$\left.\begin{aligned} (h_1-b_2)\ddot{\theta}_1+b_2\ddot{\theta}_2+g\theta_1 &= 0 \\ \text{and}\quad (h_2-b_1)\ddot{\theta}_2+b_1\ddot{\theta}_1+g\theta_2 &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (2.5)$$

On comparing these equations with those we should get for the two pendulums when not coupled, we see that the coefficient of $\ddot{\theta}_1$ for the first pendulum has been reduced by the coupling from h_1 to h_1-b_2 and the decrease b_2 made the coefficient of a term in $\ddot{\theta}_2$, while for the second pendulum the coefficient of $\ddot{\theta}_2$ has been reduced from h_2 to h_2-b_1 and the decrease b_1 made the coefficient of a term in $\ddot{\theta}_1$. As these reduced lengths appear frequently in the subsequent equations, we shall write l_1 for h_1-b_2 and l_2 for h_2-b_1, so that the equations become

$$\left.\begin{aligned} l_1\ddot{\theta}_1+b_2\ddot{\theta}_2+g\theta_1 &= 0 \\ \text{and}\quad l_2\ddot{\theta}_2+b_1\ddot{\theta}_1+g\theta_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2.5')$$

For convenience we shall suppose that $l_2>l_1$.

Multiplying the second equation by λ/b_1, where λ is a quantity to be determined, and subtracting from the first, we get

$$(l_1-\lambda)\ddot{\theta}_1+\left(b_2-\frac{\lambda l_2}{b_1}\right)\ddot{\theta}_2+g\left(\theta_1-\frac{\lambda}{b_1}\theta_2\right)=0, \quad . \quad (2.6)$$

which may be integrated directly if

$$\frac{b_2-l_2\lambda/b_1}{l_1-\lambda}=-\frac{\lambda}{b_1};$$

that is if

$$\lambda_2+(l_2-l_1)\lambda-b_1b_2=0,$$

which gives

$$\left.\begin{matrix}\lambda \\ \lambda'\end{matrix}\right\} = -\frac{l_2-l_1}{2}\pm\sqrt{\left(\frac{l_2-l_1}{2}\right)^2+b_1b_2}. \quad . \quad (2.7)$$

The integrals of the equations are then

$$\left.\begin{aligned} \theta_1-\frac{\lambda}{b_1}\theta_2&=f\sin\left(\sqrt{\frac{g}{l_1-\lambda}}.t+\phi\right)\\ \text{and}\quad \theta_1-\frac{\lambda'}{b_1}\theta_2&=s\sin\left(\sqrt{\frac{g}{l_1-\lambda'}}.t+\sigma\right), \end{aligned}\right\} \quad . \quad . \quad (2.8)$$

where f, ϕ for the faster oscillation and s, σ for the slower are constants of integration.

Solving for θ_1 and θ_2, substituting $-\frac{b_1b_2}{\lambda}$ for λ' in the amplitude terms and $l_2+\lambda$ for $l_1-\lambda'$ in the circular functions so as to retain only that root of the equation in λ which vanishes when the pendulums are not coupled, and writing F for $\frac{b_1b_2}{b_1b_2+\lambda^2}f$, S for $\frac{b_1\lambda}{b_1b_2+\lambda^2}s$, we get

$$\left.\begin{aligned} \theta_1=&\quad \mathrm{F}\sin\left(\sqrt{\frac{g}{l_1-\lambda}}.t+\phi\right)\\ &+\frac{\lambda}{b_1}\mathrm{S}\sin\left(\sqrt{\frac{g}{l_2+\lambda}}.t+\sigma\right)\\ \text{and}\quad \theta_2=&-\frac{\lambda}{b_2}\mathrm{F}\sin\left(\sqrt{\frac{g}{l_1-\lambda}}.t+\phi\right)\\ &+\mathrm{S}\sin\left(\sqrt{\frac{g}{l_2+\lambda}}.t+\sigma\right). \end{aligned}\right\} \quad . \quad (2.9)$$

Thus the four quantities F, S, ϕ, σ remain to be determined by the initial conditions of the system.

On substituting for θ_1 and θ_2 in the equations (2.2) for x_1 and x_2, we find on reduction

$$\left.\begin{aligned} x_1=&\quad \mathrm{F}'\sin\left(\sqrt{\frac{g}{l_1-\lambda}}.t+\phi\right)\\ &+\frac{\lambda}{b_1}\mathrm{S}'\sin\left(\sqrt{\frac{g}{l_2+\lambda}}.t+\sigma\right)\\ \text{and}\quad x_2=&-\frac{\lambda}{b_2}\mathrm{F}'\sin\left(\sqrt{\frac{g}{l_1-\lambda}}.t+\phi\right)\\ &+\mathrm{S}'\sin\left(\sqrt{\frac{g}{l_2+\lambda}}.t+\sigma\right), \end{aligned}\right\} \quad . \quad (2.9')$$

where $\mathrm{F}'=(l_1-\lambda)\mathrm{F}$, $\mathrm{S}'=(l_2+\lambda)\mathrm{S}$, ϕ, σ are again constants.

3. On comparing these expressions with those which would hold if the pendulums were not coupled,

$$\left.\begin{aligned} x_1 &= \mathrm{F}' \sin\left(\sqrt{\frac{g}{h_1}}.t+\phi\right) \\ x_2 &= \mathrm{S}' \sin\left(\sqrt{\frac{g}{h_2}}.t+\sigma\right), \end{aligned}\right\}$$

we note that the effect of the coupling is to replace h_1 by $l_1-\lambda$ in the case of the faster and h_2 by $l_2+\lambda$ in the case of the slower pendulum, and that each of these new oscillations is reproduced in the oscillation of the pendulum from which it is not directly derived, with an amplitude reduced in the ratio λ/b_1 in the case of the slower and $-\lambda/b_2$ in that of the faster oscillation.

With the help of this observation it is possible to write down the equations 2.9 or 2.9′ directly from inspection of fig. 1.

It will be noticed that as the signs of the amplitudes $\frac{\lambda}{b_1}\mathrm{S}$ and S or $\frac{\lambda}{b_1}\mathrm{S}'$ and S′ of the slower component oscillations of each pendulum are the same, the slower component oscillations are in phase. As those of the amplitudes F and $-\frac{\lambda}{b_2}\mathrm{F}$ or F′ and $-\frac{\lambda}{b_2}\mathrm{F}'$ of the faster component oscillations are different, the faster component oscillations are opposite in phase.

The product of the amplitudes of the component oscillations of the first pendulum is $\mathrm{FS}\lambda/b_1$, while that for the second is $-\mathrm{FS}\lambda/b_2$. The quotient of the first product by the second is $-b_2/b_1$, that is $-m_2/m_1$ *.

The ratios of the products of the maximum angular or linear velocities and of the products of the maximum angular or linear accelerations of the component oscillations of the two pendulums have also the same value, $-m_2/m_1$.

The kinetic energy T of the system is $\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2$. On substituting the values of x_1 and x_2 from the equations 2.9′ and reducing, we get

$$\mathrm{T}=\tfrac{1}{2}g\left(1+\frac{\lambda^2}{b_1b_2}\right)\left\{m_1(l_1-\lambda)\mathrm{F}^2\cos\left(\sqrt{\frac{g}{l_1-\lambda}}.t+\phi\right)\right.$$
$$\left.+\,m_2(l_2+\lambda)\mathrm{S}^2\cos\left(\sqrt{\frac{g}{l_2+\lambda}}.t+\sigma\right).\right. \quad (3.1)$$

* See Rowell, Phil. Mag. xliv. p. 382 (1922).

The potential energy V of the system is

$$\tfrac{1}{2}m_1g\{h\theta^2+(h_1-h)\theta_1^2\}+\tfrac{1}{2}m_2g\{h\theta^2+(h_2-h)\theta_2^2\},$$

which reduces in virtue of equations (2.2) and (2.9) to

$$V=\tfrac{1}{2}g\left(1+\frac{\lambda^2}{b_1b_2}\right)\left\{m_1(l_1-\lambda)F^2\sin\left(\sqrt{\frac{g}{l_1-\lambda}}\cdot t+\phi\right)\right.$$
$$\left.+m_2(l_2+\lambda)S^2\sin\left(\sqrt{\frac{g}{l_2+\lambda}}\cdot t+\sigma\right)\right\}. \quad (3.2)$$

Hence the sum of the energies

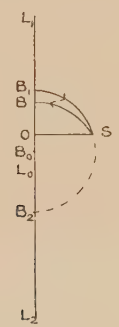
$$T+V=\tfrac{1}{2}g\left(1+\frac{\lambda^2}{b_1b_2}\right)\{m_1(l_1-\lambda)F^2+m_2(l_2+\lambda)S^2\}, \quad (3.3)$$

which is constant.

4. Before discussing special cases of the general result (2.9), it is convenient to give it a geometrical form.

In a vertical straight line (fig. 2) let B_1B_2 be taken equal

Fig. 2.



$B_1B_2=h_1$, $B_1L_1=h_1-h$, $B_2L_2=h_2-h$, $(OB_1=b_1):(OB_2=b_2)=m_1:m_2$. B_1S arc with B_0 the middle of B_1B_2 as centre, SB arc with L_0 the middle of L_1L_2 as centre, $OB=\lambda$. Component oscillations are those of BL_1 and BL_2. Amplitude ratios of first pendulum to second $=-OB_2/OB$ for faster $=OB/OB_1$ for slower components.

on some convenient scale to the length of the common portion h of each pendulum. Let B_2L_2 be the free length of one (*e. g.* the longer) and B_1L_1 that of the other pendulum. Let O divide B_1B_2 in the ratio $m_1:m_2$ so that

$$B_1O=\frac{m_1}{m_1+m_2}B_1B_2=\frac{m_1}{m_1+m_2}h$$

and

$$OB_2=\frac{m_2}{m_1+m_2}B_1B_2=\frac{m_2}{m_1+m_2}h.$$

Through O draw the horizontal OS. With B_0, the middle point of B_1B_2 as centre, describe a circular arc cutting the horizontal through O in S. Then $OS=\sqrt{b_1b_2}$. With L_0 the middle point of L_1L_2 as centre, describe a circular arc SB cutting OB_1 in B. Then, since $OL_1=l_1$ and $OL_2=l_2$,

$$OL_0=\frac{l_2-l_1}{2} \text{ and } BL_0=L_0S=\sqrt{\left(\frac{l_2-l_1}{2}\right)^2+b_1b_2}.$$

Thus

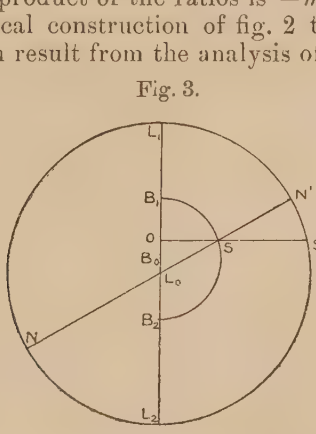
$$OB=\sqrt{\left(\frac{l_2-l_1}{2}\right)^2+b_1b_2}-\frac{l_2-l_1}{2}=\lambda,$$

$BL_1=l_1-\lambda$ and $BL_2=l_2+\lambda$, so that BL_1 and BL_2 are the lengths of the simple pendulums which have times of oscillation equal to those of the faster and slower component oscillations from which the actual oscillations of the two pendulums are compounded.

The ratio of the amplitude of the faster component oscillation of the first pendulum to that of the second is $-b_2/\lambda$, that is $-OB_2/OB$; while the ratio of the amplitude of the slower component of the first to that of the second is λ/b_1, that is OB/OB_1. The product of the ratios is $-m_2/m_1$.

The geometrical construction of fig. 2 thus gives all the quantities which result from the analysis of section 2.

Fig. 3.



Component oscillations those of N'S and SN. Coupling $\gamma=\frac{OS}{OS'}$.

When the periods of the component oscillations only are required, it is simpler after finding S to join it to L_0 and produce to cut the circle on L_1L_2 as diameter in N and N' (fig. 3). Then SN and SN' are the lengths of the simple pendulums which give the component oscillations.

If, in fig. 3, OS be produced to cut the circle on L_1L_2 as diameter in S', the ratio $OS/OS'=\sqrt{b_1b_2/l_1l_2}$ is what Barton

and Browning propose *, in analogy with the terminology in use in connexion with coupled electric circuits, to call the "coupling γ" of the two pendulums.

As the length λ plays so important a part in the determination of the oscillations of the system, it would be useful to give it a name—*e.g.*, the "linking" or "key" length.

5. Consideration of special cases.

Case 1.—The lengths of the pendulums are equal and their masses equal. This is the simplest case. For it, O, B_0, and L_0 all coincide and B coincides with B_1. Thus the periods of the component oscillations are those of pendulums of lengths B_1L_1 and B_1L_2, that is h_1-h and h_2, and the ratio of the amplitudes of the component oscillations of the first pendulum to that of the second are $-B_0B_2/B_0B_1=-1$ for the faster and $B_0B_1/B_0B_1=1$ for the slower. The values of F and S, ϕ and σ depend on the initial conditions of the motion. If these conditions are such that F=S, each pendulum in turn comes to rest when the other has its oscillation † of maximum amplitude.

Case 2.—The lengths of the pendulums unequal, but their masses equal. In this case O and B_0 of fig. 2 coincide, but B is not identical with B_1. The amplitude ratios of the first pendulum to the second are for the faster component oscillation $-B_0B_2/B_0B$ and for the slower B_0B/B_0B_1 which are numerically reciprocals. If the initial conditions are such as to make the amplitudes of the faster and slower component oscillations of one pendulum equal, that pendulum comes to rest while the other has its oscillation of maximum amplitude; but we no longer have each pendulum in turn coming to rest while the other has its oscillation of maximum amplitude ‡.

Case 3.—With the masses m_1 and m_2 unequal but the lengths h_1 and h_2 equal we have l_1 and l_2 unequal, and in fig. 2 B_0 and L_0 coincide; the two circles are concentric, and therefore B and B_1 coincide. Thus the amplitude ratio of the first to the second pendulum is for the faster component oscillation $-OB_2/OB_1$ and for the slower component oscillation $OB_1/OB_1=1$. The initial conditions might be such as to make the two component oscillations of one pendulum equal in amplitude, but those of the other would then be unequal.

* Phil. Mag. xxxiv. p. 254 (1917).

† See Barton & Browning, Phil. Mag. xxxiv. p. 255 (1917) and the curves given by them for this case.

‡ Barton & Browning, Phil. Mag. xxxv. p. 71 (1918), equations (69) and (70) and the curves 20–28 illustrating such cases.

It would be possible to have one pendulum coming to rest when the amplitude of the oscillation of the other was a maximum, but the other would not come to rest *.

Case 4.—In the general case in which neither the masses m_1 and m_2 nor the lengths h_1 and h_2 are equal, l_1 and l_2 are unequal, and in fig. 2 O, B_0, and L_0 are distinct points. B does not coincide with B_1, the amplitude ratios are unequal, and it would again be possible by adjusting the initial conditions to secure that one only of the pendulums should have the same amplitude for each of its component oscillations. Hence only one of the two pendulums could have zero amplitude at the time that the other had its oscillation of maximum amplitude †.

It is possible that with inequality of the masses m_1 and m_2 and inequality of the lengths h_1 and h_2 we may have $l_1 = l_2$. In that case O and L_0 of fig. 2 coincide, but B_0 does not coincide with either, nor does B with B_1. Thus the amplitude conditions are those of the general case.

6. The effect of changing the value of one of the variables m_1, m_2, h_1, h_2, and h may be traced readily by reference to fig. 2. Thus when m_1 is very small compared with m_2, OB_1 is small compared with OB_2, OS is small, and B close to B_1 and between B_1 and O if l_2 is greater than l_1. The faster component oscillations correspond to those of a pendulum of length slightly greater than $h_1 - h$ the free length of the shorter pendulum, and the slower oscillations to those of a pendulum of length slightly less than h_2 the total length of the longer pendulum.

As the mass m_1 increases in proportion to m_2, OB_1 increases at the expense of OB_2, OS increases, and B separates more from B_1. The faster component oscillation therefore slows down and the slower speeds up, and when the two masses are equal, the faster oscillation is that of a pendulum of length

$$\frac{h_1+h_2-h}{2} - \sqrt{\left(\frac{h_2-h_1}{2}\right)^2 + \left(\frac{h}{2}\right)^2},$$

and the slower that of one of length

$$\frac{h_1+h_2-h}{2} + \sqrt{\left(\frac{h_2-h_1}{2}\right)^2 + \left(\frac{h}{2}\right)^2}.$$

* Compare Barton & Browning, Phil. Mag. xxxv. p. 66 (1918), equations (27) and (28) (which, however, include damping) and the curves 1–19 illustrating these cases.

† Compare Barton and Browning, Phil. Mag. xxxvi. p. 40 (1918), equations (21) and (22) (which, however, include damping) and the curves in illustration.

If m_1 increases still further, OB becomes greater than OB_2, and the slowing down of the faster and the speeding up of the slower component oscillations continues. When O coincides with L_0, $l_1 = l_2$.

It will be noticed that the amplitude ratios are not affected by a change in the distances B_1L_1, B_2L_2 of fig. 2 so long as L_0, the middle point of L_1L_2, remains fixed. Hence the lengths of the independent portions of the pendulums may be increased or decreased each by the same amount without change of the amplitude ratios given by OB, OB_1, and OB_2. In particular, B_1L_1 may be made zero, and fig. 2 then gives the solution of the problem of the oscillations of a system consisting of one simple pendulum suspended from the bob of another simple pendulum.

7. From the preceding paragraphs it will have been seen that the whole of the dynamics of coupled pendulums of the type here considered may be represented by the graphical construction of fig. 2, which entails the drawing of two straight lines and two arcs of circles. It determines the periods of the faster and slower component oscillations, the ratio of the amplitudes of the faster components and the ratio of the amplitudes of the slower components. To complete the solution of any particular case, we require in addition the actual values of the amplitudes and one of the faster and one of the slower component oscillations and the two phase angles. This further information may be furnished by the initial conditions or by the conditions at some other time or times.

In all cases in which the periods of the component oscillations are commensurate, it is possible to find an instant t_0 from which to reckon the time, which will rid our equations of the phase angles. Thus, if t_0 satisfies the equations

$$\sin\left(\sqrt{\frac{g}{l_1-\lambda}}\cdot t_0+\phi\right)=\sin\left(\sqrt{\frac{g}{l_2+\lambda}}\cdot t_0+\sigma\right)=0, \quad (7.1)$$

that is

$$t_0=\frac{n\pi-\phi}{\sqrt{\frac{g}{l_1-\lambda}}}=\frac{n'\pi-\sigma}{\sqrt{\frac{g}{l_2+\lambda}}},$$

where n and n' are suitable integers, and we reckon time t from the instant t_0, the equations for the angular oscillations

of the pendulums take the form

$$\left.\begin{aligned}\theta_1 &= \quad \mathrm{F}\sin\sqrt{\frac{g}{l_1-\lambda}}\cdot t+\frac{\lambda}{b_1}\mathrm{S}\sin\sqrt{\frac{g}{l_2+\lambda}}\cdot t,\\ \theta_2 &= -\frac{\lambda}{b_2}\mathrm{F}\sin\sqrt{\frac{g}{l_1-\lambda}}\cdot t+\quad \mathrm{S}\sin\sqrt{\frac{g}{l_2+\lambda}}\cdot t,\end{aligned}\right\} \quad (7.2)$$

with corresponding equations for the linear displacements of the bobs of the two pendulums.

In either case it is usual to represent the compound oscillation of each pendulum by means of a curve with time as abscissæ and the angular or linear displacement as ordinates. The shape of each of these curves is generally of greater interest than the absolute values of the ordinates, so that the ratio of the amplitude of the faster to that of the slower component oscillation of each pendulum is sufficient for most purposes. This ratio may have values ranging from $-\infty$ to $+\infty$ in different cases, but it is convenient in drawing the curves to take each as having the same maximum ordinate, say 1, that is the sum of the amplitudes of the quicker and slower component oscillations in each case = 1, and therefore each of the amplitudes may vary in magnitude between 0 and 1. The gradual change of shape of the compound oscillation as the ratio of the amplitudes of the two component oscillations changes from 0 to ∞ may be sufficiently illustrated by taking five intermediate stages between the extreme cases in which either component is present alone.

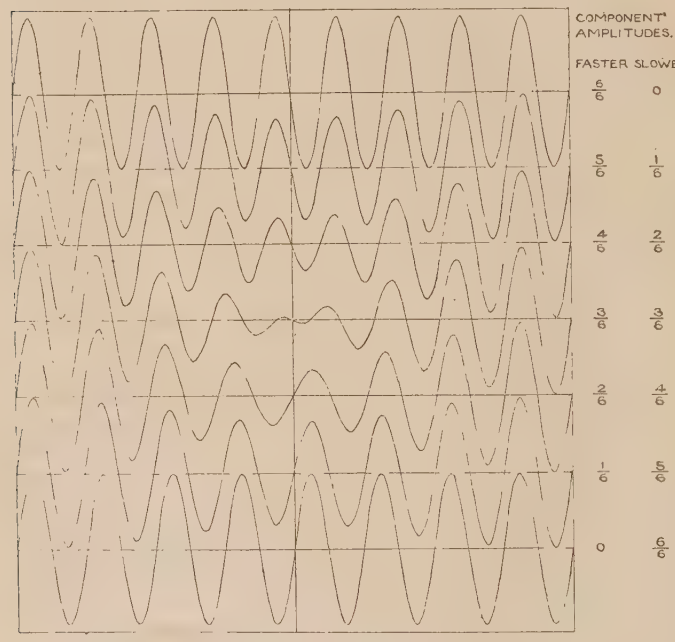
8. The periods of the component oscillations may bear any ratio to each other, but in a great number of the most interesting cases the ratio approximates to that of two consecutive integers exceeding 4 or 5 to each other. The compound oscillations in all these cases show the same principal features, and may be represented by a typical example. In fig. 1 the lengths of the two pendulums are as 13 : 15, their common part 8 and their masses as 3 : 5. A more interesting case is that of two pendulums of lengths $h_1=40$ and $h_2=42$ with a common length $h=8$, the ratio of the masses 3 : 5, so that fig. 2 still holds (section 6) if B_1L_1 and B_2L_2 are each increased by 27 units. In this case $b_1=3$, $b_2=5$, $l_1=35$, $l_2=39$, $l_2-l_1=4$, $\lambda=2{\cdot}36$, $l_1-\lambda=32{\cdot}64$, $l_2+\lambda=41{\cdot}36$, the ratio of the frequencies of the component oscillations is 9/8 and the coupling (sect. 4) 0·104, *i.e.* 10·4 per cent.

The curves of fig. 4 show the results of compounding two oscillations of frequencies in this ratio with amplitudes of

the faster and the slower oscillations respectively 1 and 0, 5/6 and 1/6, 4/6 and 2/6, 3/6 and 3/6, 2/6 and 4/6, 1/6 and 5/6, 0 and 1. In each curve the two component oscillations are in the same phase at the left-hand and right-hand margins, and opposite in phase at the middle line of the diagram.

If we assume that in a particular case the initial conditions are such that for the pendulum h_1 at the epoch from which time is measured, the ratio of the amplitudes of fast to slow component is 5 to 1 and the two components are in phase,

Fig. 4.



Oscillations compounded of two components of frequencies as 9 : 8, and amplitudes stated.

the second curve from the top of the diagram represents the compound oscillation. But this fixing of the ratio $F'/\frac{\lambda}{b_1}S'$ of the amplitudes of the components for one pendulum also fixes it for the other, for which, according to equation (2.9), it must have the value $-\frac{\lambda}{b_2}F'S'$, that is $-\lambda^2/b_1b_2$ times the former, or $-0{\cdot}371\times 5=-1{\cdot}85$. The compound oscillation of the second pendulum is therefore represented by a curve nearly identical with the third curve from the top of the diagram.

But since the component oscillations are opposite in phase at the epoch from which time is reckoned, that part of the curve for the second pendulum between the middle line and the right-hand margin of the diagram corresponds to that between the left-hand margin and the middle line in the case of the first pendulum.

Assuming any other ratio for the amplitudes of the component oscillations of one pendulum, we fix in the same way the ratio for the other, and in each case select from the curves of fig. 4 those which represent most nearly the compound oscillation of each pendulum.

For accurate work a set of curves would be necessary for each value of the ratio of the frequencies of the component oscillations ; but so long as the ratio is that of two consecutive integers not less than 4, the principal features of the various sets are the same, and fig. 4 illustrates sufficiently the insight the graphical method of fig. 2 is capable of giving into the problems of coupled pendulums in cases where the damping is negligible.

XV. *Dimensional Analysis.* *By* Edgar Buckingham *.

In an interesting article under this heading in the March number of the Philosophical Magazine, Dr. Norman Campbell expresses some dissatisfaction with the manner in which the late Lord Rayleigh's method of reasoning from dimensions has been used, or abused, in recent years. I agree, in substance, with much of what he says, and yet I think the final impression he leaves is false because he exaggerates the unimportance, uselessness, and unreliability of the method. Physicists who have learned from experience the value of dimensional reasoning, the principle of similarity, or whatever else it may be called, will not be much affected by arguments either for or against it. But since, if Dr. Campbell's article passed without comment, those who have not acquired this familiarity might be led to suppose that the subject was not worth cultivating, it seems well to note that some other points than those particularly emphasized by Dr. Campbell also deserve attention.

As regards the question of utility—that is a purely practical matter to be settled by experience. It is much like the question whether it is better to use a slide-rule, a log table,

* Communicated by the Author. Published by permission of the Director of the Bureau of Standards of the U. S. Department of Commerce.

or longhand computation : it depends on circumstances and also partly on taste or habit. But that the method has been very useful can be proved beyond a doubt without going outside the limits of London. One has only to examine the reports of the National Physical Laboratory and the brilliant record of experimental work by the Staff of the Engineering and Aerodynamics Departments to be convinced that dimensional analysis—to give it its most convenient name, in despite of Dr. Campbell—has been a very valuable tool. The members of the N. P. L. have not, to be sure, *talked* much about the method : they have been too busy with interesting and difficult experiments to spend time in explaining the details of what seemed to them simple and obvious. But they have *used* it constantly ; and if they have ever been led astray by it, I do not remember their saying so.

The contemplation of the very fine series of investigations just referred to, leads naturally to the question what sort of things dimensional analysis is most useful for ; and in the answer there is, I think, one main reason for the difference between Dr. Campbell's feeling toward the subject and mine.

The dimensional method has often been employed for the purpose of deducing new relations in pure physics without going through more laborious investigations of which it was evident beforehand that the results would involve unknown constants. This is the way in which Lord Rayleigh used the method most, a good example being his λ^4 relation for scattered light. I suspect that this is the sort of problem in which Dr. Campbell is most interested, but it is the one in which it is easiest to make mistakes. And if he can catch Jeans or Einstein in a slip, the rest of us may enjoy the joke without any malice, for they can afford to slip occasionally.

But the principal value of dimensional analysis is not for obtaining new theoretical results or for proving anything *a priori*. No new result can be proved, to demonstration, except by experiment; for theoretical physics, of whatever sort, can only say that *if* so and so is true, and *if* we have not overlooked anything, then something else must also be true. As long as there is an *if* anywhere, the final decision rests with experiment. The main value of dimensional analysis is as a guide in planning experimental work and, particularly, in analysing and interpreting the observational results. The more complicated the problem and the greater the number of variables involved, the more useful does such a guide become ; and this remark leads directly to the consideration of another point in Dr. Campbell's paper.

He deplores the eruption of ill-digested dimensions in the

last few years, and seems to imply that some degree of responsibility for this state of things rests on those who, being convinced of the utility of the dimensional method, have attempted to expound it in an elementary textbook style, so as to induce others to try it. But this reproach, if intended, is wide of the mark ; for it is perfectly obvious that the immediate occasion for the recent great increase in the use of dimensional arguments is the complete demonstration of their value by the Staff of the N. P. L., which has led others to take up the method until now it is used everywhere.

There is, however, a more fundamental reason for the increased popularity of the dimensional method. Lord Rayleigh used it in 1871 in one of his earliest papers, he was interested in it all his life, and he employed it freely in numerous cases; and yet no one else had used it much until recently. Why is this? The answer is, clearly, that technical physics, the physics of engineering and manufacturing processes, has had an immense development in the last dozen years, and that it is precisely in technical physics that dimensional analysis is of most value and importance.

In the experimental researches in pure physics, with which most physicists are occupied, it is usually possible so to control conditions that the theory of the experiment shall be simple. The number of variables can be kept small, it is easy to write down the differential equations, and—this is the important point—the experiment can be so arranged that the equations which describe it can be solved without great labour. When this is impossible, the investigator who is free to suit himself is likely to turn to some more tractable problem, for there are still plenty of them.

But in technical physics we find a very different state of affairs. Problems which are clamouring for solution must be taken up as they are, with all their complications, and cannot be thrown aside merely because they are difficult. It is often not possible to isolate each separate phase of the problem, investigate it by itself, and then get the general solution by synthesis. It may be a difficult matter to write down any equation or set of equations for describing the phenomenon in question; and even if equations have been obtained, it is usually obvious that the task of solving them would be impracticably laborious and time-consuming, if not altogether impossible. It is just in such cases that the rough and ready method of dimensions comes in handy and proves itself very much better than nothing at all.

No one, of course, will think of denying that when ordinary methods will work, the definite results to which they lead are

worth more than anything that can be got from dimensional analysis alone; but the question is: "is half a loaf better than no bread?" and not whether it is better than a whole cake. And even here a counter attack may be made. For, as Professor W. F. G. Swann once remarked: it often happens that when, by a long mathematical process, we have obtained some formidable combination of symbols with a numerical coefficient, the correct value of the coefficient remains doubtful and the only part we are sure is right might just as well have been obtained by dimensions. There is truth in this remark along with its exaggeration.

Dr. Campbell's main contentions are, I take it, 1st that we can never be certain that our dimensional treatment of any problem is correct unless we have actually written down the differential equations; 2nd that if we have done this, we can get all the information obtainable from the principle of dimensional homogeneity by mere inspection of the equations, so that all cumbersome algebraic machinery, such as that of the Π theorem, is superfluous; and 3rd that, being both cumbersome and superfluous, it should be buried and forgotten.

To the first of these it may be remarked that we are never absolutely sure of anything in physics till it has been proved by experiment, even if we have written down a differential equation; and that in almost any operation, avoiding mistakes is a matter of care, experience, and skill. A lawyer trying to drive a nail may hit his fingers, whether he uses a hammer or a hatchet: an expert carpenter is less likely to do so, and he will use whichever tool is the more convenient in view of the other things he may be having to do in connexion with driving the nail.

The second point may be granted without discussion,—the carpenter's hammer is not indispensable; but it is a very useful tool even though one cannot chop with it.

The third point may be left for the future to decide. I think that beginners will find the Π theorem a good thing to cut their teeth on; but in any event, after one has reached the stage in which all the results obtainable by the formal algebra are evident upon mere inspection, one need no longer pay any attention to it. I came upon the Π theorem in the course of trying to follow the reasoning in some of the papers in the early volumes of the N. A. C. A. Reports, and found it advantageous to have the theorem stated in some detail; but the form in which the dimensional argument is put is of small importance: what counts is its utility to people who know how to use it, and that is indisputable. In reference to the question how Lord Rayleigh would have felt about the

formal discussion of the dimensional method, I may quote from a letter of October 30, 1915, in which he wrote, regarding my first attempt, in the 'Physical Review' of Oct. 1914: "It is the best exposition of the subject I have come across; the only doubt is whether it is not too good for a majority of Physicists!" And three months later, after some correspondence: "I hope you will continue your work on this subject."

To anticipate all the possible mistakes that might be made in the name of dimensions would require omniscience, but many physicists of quite ordinary endowments have found the method of dimensions so useful that they accept the risk with equanimity. Their salvation lies not in formal logic, not in differential equations, and not in arguments about the dimensions of particular quantities which may always be stated in various ways, but, as Dr. Campbell implies, in physical common sense.

Washington,
April 8, 1924.

XVI. *Note on a Search for the missing Element No. 43.* *By* C. H. Bosanquet *and* T. C. Keeley *.

THE smallest atomic number corresponding to an element not yet detected is 43, an element which from its place in the periodic table should be a metal resembling Manganese in its reactions.

During the past year a number of manganese minerals have been investigated in hopes of detecting this element, as it was considered possible that amounts of the order of 1 per cent. might have escaped detection by chemical methods in such minerals. The results have, so far, been negative.

The method used in the investigation was to photograph the X-ray spectrum of each mineral between the wave-lengths 620 and 720 X.U.

Experimental.

The tube used had an anti-cathode whose face was horizontal and normal to the cathode-ray stream, and viewed edgeways this constituted a line source of X-rays sufficiently sharp to allow one to dispense with collimator slits.

* Communicated by Prof. F. A. Lindemann, F.R.S.

A piece of calcite with a face about 4 cm. long. in the direction of the rays served as grating; it was fixed in position about 25 cm. from the anti-cathode and equidistant from the anti-cathode and the plate-holder, a lead screen with a slit 6 mm. wide placed immediately in front of the crystal serving to protect the plate from direct radiation. The crystal was a very perfect specimen, and gave no trouble through irregularities of the face, so that it was unnecessary to rock it. Adjustment was tested by photographing the K spectrum of molybdenum (42) which just covers the range required, the crystal being large enough to take in the whole spectrum at one setting. The Kα line of 43 would occur approximately midway between the Kα and Kβ lines of molybdenum. The apparatus gave a dispersion of approximately 10 X.U. per mm., the resolving power being always sufficient to separate the Kα doublet of molybdenum.

The minerals, in some cases mixed with a small quantity of Mo filings, were fused with the oxyhydrogen flame into small nickel cups lined with alundum cement, and then ground down flush with the edges of the cups. The cups were screwed into the removable anti-cathode holder, which located them sufficiently exactly to ensure the Kα line of 43 coming into the field.

The exposures were never less than 1 milliampere hour at 50 k.v., which was sufficient to show the presence of 0·2 per cent. of added Mo, and in several cases, when the tube was running well, the exposures were much longer than this.

In most cases it was necessary to pass a discharge for several hours with the pump working before it was possible to maintain a potential difference of as much as 40 k.v. across the tube, but when the tube had become sufficiently hard to give off Mo rays in considerable quantities, it passed in an hour or so through several stages of increasing hardness. In the final stages it was possible to shut off the pump completely, and eventually air had to be admitted at frequent intervals in order to keep the tube sufficiently soft. This was very conveniently accomplished by using a good tap with a long lever fixed to it to allow of fine adjustment; the tap communicated directly with the atmosphere.

The current which could be used varied, according to the mineral used and the state of the tube, between 1 and 0·2 m.a., the anti-cathode being maintained at a fairly bright red heat.

Minerals used.

The origin of the minerals used is given in the second column; in a few cases no information was available on this point.

Mineral.	*Origin.*
Psilomelane.	Broken Hill, Australia.
Psilomelane.	Bohemia.
Rhodonite.	Perm, Russia.
Rhodonite.	Chikhla, Central Prov., India.
Rhodochrosite.	Hungary.
Rhodochrosite.	Park Co., Col., U.S.A.
Polyanite.	Upton Pike (Exeter ?).
Pyrolusite.	Davidson's Mine, King's Co.
Pyrolusite.	Newton Down (Devon ?).
Wad.	Bohemia.
Manganite.	Unknown origin.
Franklinite.	
Pyrolusite.	
Psilomelane.	

In addition to these minerals there were also examined a specimen of Mn extracted from Torbernite supplied by Dr. A. S. Russell, the mixed Ni, Co, and Mn extracted from Thorianite supplied by Professor T. R. Merton, and a specimen of mixed metals from the Urals supposed to contain Ru (44), Rh (45), and Pd (46), supplied by Mr. A. C. E. Egerton.

In addition to those mentioned above our thanks are due to Mr. T. V. Barker and Mr. E. J. Bowen, who supplied us with the minerals used; also to Professor F. A. Lindemann, F.R.S., for his interest and advice during the course of the work.

XVII. *Experiments on the Duration of Impacts, mainly of Bars with Rounded Ends, in elucidation of the Elastic Theory.* *By* J. E. P. Wagstaff, *M.A., Fellow of St. John's College, Cambridge, Lecturer in Physics in the University of Leeds* *.

IN a recent paper †, an expression of the form

$$t=(\mathrm{M}r^2/a+\mathrm{B}_1 l)v^{\gamma}(83{\cdot}3\rho^{1/2}/\mathrm{E}^{1/2})^{\gamma+1},$$

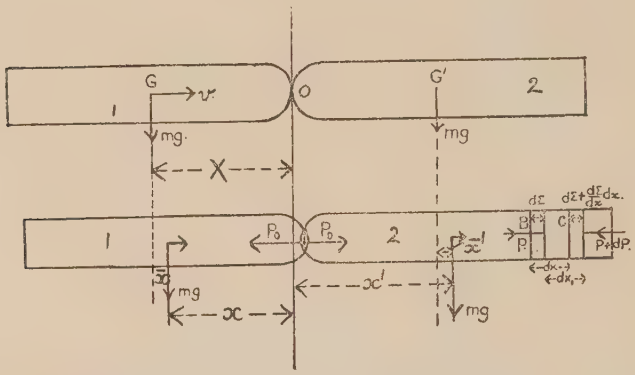
is derived by the method of dimensions for the duration of collision of two perfectly elastic bars. It is now proposed to discuss the form of the mathematical relationships which can account for an expression of this type.

* Communicated by the Author.

† Proc. Roy. Soc. A. vol. cv. (1924).

Take a system of rectangular axes at O (fig. 1), OG and OG′ being the axes of x for the left-hand bar and the right-hand bar respectively. In the lower figure, the bodies are in a state of compression, the centres of gravity G and G′ having moved through distances x and x' respectively.

Fig. 1.



If P_0 is the pressure between the bodies at this instant, the equations of motion of the two bars are

$$P_0 = m_0 \frac{d^2x}{dt^2}, \quad \text{and} \quad P_0 = m_0 \frac{d^2x'}{dt^2},$$

where m_0 is the mass of either bar.

$$\therefore \quad 2P_0 = m_0 \frac{d^2}{dt^2}(x + x').$$

If X is the distance of G and G′ from O initially,

$$x = X - \bar{x}, \quad x' = X + \bar{x}',$$

therefore $$x + x' = 2X - (\bar{x} - \bar{x}'),$$

so that $$2P_0 = -m_0 \frac{d^2}{dt^2}(\bar{x} - \bar{x}').$$

$\bar{x} - \bar{x}'$ represents the change in the relative position of the centres of gravity of the two bars and may be denoted by α. The equation thus becomes

$$2P_0 = -m_0 \frac{d^2}{dt^2}\alpha. \qquad \text{(I.)}$$

In order to obtain a further relationship between P_0 and α, consider the portion of the bar between planes distance dx

apart initially (fig. 1). If the section at B is displaced a distance Σ_x at the instant the pressure there is P, the section at C is displaced $\Sigma_x + \frac{d\Sigma_x}{dx}.dx$, the corresponding pressure being $P+dP$.

The equation of motion of the layer is

$$-\frac{dP}{dx}.dx_1 = \rho_1 A_1 dx_1 \frac{d^2\Sigma}{dt^2},$$

and $A\rho dx = A_1\rho_1 dx_1$, where A, ρ, dx refer to the cross-section, density, and thickness of the layer in the undisturbed state and the suffixed letters to the corresponding quantities in the disturbed state.

Further assume that at each point of the bar,

$$\frac{d\Sigma}{dt} = VF\left(\frac{d\Sigma}{dx}\right),^*$$

then

$$P = \rho A V^2 \int \frac{\left\{F'\left(\frac{d\Sigma}{dx}\right)\right\}^2}{1+\frac{d\Sigma}{dx}} d\left(\frac{d\Sigma}{dx}\right) = \rho A V^2 \psi\left(\frac{d\Sigma}{dx}\right).$$

In general, an equation of this type indicates a disturbance which is continuously changing its form with time.

If $F\left(\frac{d\Sigma}{dx}\right) = \frac{d\Sigma}{dx}$, then $\frac{d\Sigma}{dt} = -V\frac{d\Sigma}{dx}$, which is a disturbance of invariable form represented by $\Sigma = \phi(x - Vt)$ propagated with velocity V.

In particular assume $\psi\left(\frac{d\Sigma}{dx}\right) = \left(\frac{d\Sigma}{dx}\right)^\beta$ and examine the results to which it leads,

$$-P/A = \rho V^2 \left(\frac{d\Sigma}{dx}\right)^\beta = k\left(\frac{d\Sigma}{dx}\right)^\beta.$$

Such a pressure condensation relationship does not imply any departure from a condition of perfect resilience.

The equation $-P/A = k\left(\frac{d\Sigma}{dx}\right)^\beta$, which it is suggested is the form of the stress-strain relationship in the case of impacts of the kind that are here being considered, has the

* Earnshaw, Phil. Trans. p. 146 (1859).

same type of relation to Hooke's law $-\mathrm{P/A}=k\frac{d\Sigma}{dx}$ as the gaseous law for adiabatic changes $p=\kappa\rho^{\gamma}$ has to the isothermal equation $p=\kappa\rho$.

Moreover the centre of gravity of the layer is displaced from the origin a distance

$$\Sigma_x+\frac{d\Sigma_x}{dx}\cdot\frac{dx}{2}.$$

The centre of gravity of the right-hand bar is consequently moved through a distance $\bar{x}'$ given by

$$\bar{x}'=\frac{\rho}{m_0}\int_0^l \mathrm{A}\left(\Sigma_x+\frac{d\Sigma_x}{dx}\cdot\frac{dx}{2}\right)dx$$

away from the origin O.

If $$\mathrm{P/A}=-k\left(\frac{d\Sigma}{dx}\right)^{\beta},$$

$$\Sigma_x-\Sigma_0=-\int_0^x(\mathrm{P/A}k)^{1/\beta}dx,$$

and $$\bar{x}'=\frac{\rho\mathrm{A}}{m_0}\int_0^l dx\left(\Sigma_0-\int_0^x(\mathrm{P/A}k)^{1/\beta}dx\right).$$

Similarly for the left-hand bar,

$$\bar{x}=\frac{\rho\mathrm{A}}{m_0}\int_0^l dx\left(\Sigma_0+\int_0^x(\mathrm{P/A}k)^{1/\beta}dx\right),$$

and

$$\alpha=\bar{x}-\bar{x}'=\frac{2\rho\mathrm{A}}{m_0(\mathrm{A}k)^{1/\beta}}\int_0^l dx\int_0^x \mathrm{P}^{1/\beta}dx.$$

If the pressure P is expressible as a function of x/l, $\mathrm{P}^{1/\beta}=f(x/l)$, the value of the integral

$$\begin{aligned}\int_0^x dx\int_0^x \mathrm{P}^{1/\beta}dx&=\int_0^x dx\int_0^x f(x/l)dx,\\&=l^2\int_0^t dt\int_0^t f(t)dt \quad \text{where } x/l=t,\\&=l^2\left(\int_0^t\{\phi(t)-\phi(0)\}dt\right) \quad \text{where } \frac{\partial\phi}{\partial t}=f,\\&=l^2(\psi(t)-\psi(0)-t\phi(0)) \quad \text{where } \frac{\partial\psi}{\partial t}=\phi.\end{aligned}$$

Again,

$$\psi(t)=\psi(0)+t\psi'(0)+t^2\psi''(0)/2!+t^3\psi'''(0)/3!+\ldots\ldots;$$

therefore

$$l^2(\psi(t)-\psi(0)-t\phi(0))=l^2\left(t^2f(0)/2!+t^3\left(\frac{\partial f}{\partial t}\right)_0\Big/3!+\ldots\ldots\right),$$

$$=x^2\mathrm{P}_0^{1/\beta}/2!+x^3\left(\frac{\partial}{\partial x}\mathrm{P}^{1/\beta}\right)_0\Big/3!+\ldots\ldots,$$

i.e.
$$\int_0^l dx\int_0^x \mathrm{P}^{1/\beta}dx=l^2\mathrm{P}_0^{1/\beta}/2!+l^3\left(\frac{\partial}{\partial x}\mathrm{P}^{1/\beta}\right)_0\Big/3!+\ldots\ldots.$$

Now $\left(\frac{\partial}{\partial x}\mathrm{P}^{1/\beta}\right)_0$, $\left(\frac{\partial^2}{\partial x^2}\mathrm{P}^{1/\beta}\right)_0$ and all subsequent differential coefficients are extremely small. For

$$\left(\frac{\partial}{\partial x}\mathrm{P}^{1/\beta}\right)_0=\frac{1}{\mathrm{V}}\left(\frac{\partial}{\partial t}\mathrm{P}^{1/\beta}\right)_0 \quad \text{where} \quad \left(\frac{\partial}{\partial t}\mathrm{P}^{1/\beta}\right)_0$$

is the rate of variation of the pressure with time at $x=0$, and V is the velocity of sound. If, as Hertz assumes, impacts are relatively slow processes, $\left(\frac{\partial}{\partial t}\mathrm{P}^{1/\beta}\right)_0$ is small and $\left(\frac{\partial}{\partial x}\mathrm{P}^{1/\beta}\right)_0$ is very much smaller, and similarly for the higher order differential coefficients.

Therefore

$$\int_0^l dx\int_0^x \mathrm{P}^{1/\beta}dx=l^2\mathrm{P}_0^{1/\beta}/2,$$

and

$$\alpha=\frac{2\rho\mathrm{A}}{m_0(\mathrm{A}k)^{1/\beta}}\cdot\frac{\mathrm{P}_0^{1/\beta}l^2}{2}.$$

Writing
$$m_0=\pi r^2 l\rho,$$

$$\mathrm{P}_0=k\pi r^2(\alpha/l)^\beta. \quad \ldots\ldots\ldots \quad \text{(II.)}$$

Substitute for P_0 in (I.),

$$2k(\alpha/l)^\beta\pi r^2=-m_0\frac{d^2\alpha}{dt^2},$$

multiply by $2\frac{d\alpha}{dt}$ and integrate

$$\left(\frac{d\alpha}{dt}\right)^2+\frac{2k\pi r^2}{m_0 l^\beta}2\cdot\frac{\alpha^{\beta+1}}{\beta+1}+\text{constant}=0.$$

Initially $\alpha=0$ and $d\alpha/dt=v$, therefore $v^2+\text{constant}=0$,

i.e.
$$\frac{\left(\frac{d\alpha}{dt}\right)^2-v^2}{2}+\frac{2k\pi r^2}{m_0 l^\beta}\cdot\frac{\alpha^{\beta+1}}{\beta+1}=0. \quad \ldots \quad \text{(III.)}$$

At the moment of greatest compression $\frac{d\alpha}{dt}=0$. If the value of α is then α_1,

$$\frac{2k\pi r^2}{m_0 l^\beta}\cdot\frac{{\alpha_1}^{\beta+1}}{\beta+1}=v^2/2.$$

Integrating (III.),

$$\frac{d\alpha}{\left(v^2-\frac{4k\pi r^2\alpha^{\beta+1}}{m_0 l^\beta(\beta+1)}\right)^{1/2}}=dt,$$

and the time of impact is given by

$$t=2\int_0^{\alpha_1}\frac{d\alpha}{\left(v^2-\frac{4k\pi r^2\alpha^{\beta+1}}{m_0 l^\beta(\beta+1)}\right)^{1/2}}$$

$$=\frac{2}{v}\left(\frac{v^2 l^\beta m_0(1+\beta)}{4k\pi r^2}\right)^{1/1+\beta}\int_0^1\frac{dx}{(1-x^{\beta+1})^{1/2}}.$$

Writing $m_0=\pi r^2 l\rho$ and $\gamma=1+\beta/1-\beta$,

$$t=l\pi^{1/2}v^\gamma\frac{(2/1+\gamma)^{1+\gamma/2}\Gamma(1+\gamma/2)}{4^{1+\gamma/2}\Gamma(2+\gamma/2)}(\rho/k)^{1+\gamma/2}, \quad . \quad \text{(IV.)}$$

which is of precisely the same form as that obtained by the method of dimensions.

Assuming that for very long rods $\gamma\to 0$, the time of collision reduces to $t=2\cdot 2l(\rho/k)^{1/2}$. Since $-1/\gamma$ has been found to be expressible in the form $-1/\gamma=A''+Kl/r$,

$$\beta=\frac{A''+1+Kl/r}{A''-1+Kl/r}=\frac{P+Kl/r}{Q+Kl/r},$$

so that on substitution in equation (II.),

(*a*) for long rods $\beta\to 1$, and $P_0=k\pi r^2(\alpha/l)$,
(*b*) for short rods $\beta\to P/Q$, and $P_0=k\pi r^2(\alpha/l)^{P/Q}$,
(*c*) for intermediate rods

$$\beta=\frac{P+Kl/r}{Q+Kl/r} \text{ and } P_0=k\pi r^2(\alpha/l)^{\frac{P+Kl/r}{Q+Kl/r}}.$$

In (*a*) the stress strain relationship reduces to Hooke's law, so that the stress can be assumed to be distributed uniformly over the section of the bar; the point source of pressure has lost its identity and the various elements of the original wave have gathered themselves up into a plane wave (compare with the case of a liquid injected into a long narrow tube through a point source at one end).

In (*b*), on the other hand, the cylindrical boundary exerts no powerful influence on the impact owing to the shortness of the rods. The pressure becomes dissipated through the body and the various elements are not able to reinforce one another and produce a combined wave-front. The rapid dispersion of the stress also explains why, in the spherical problem, Hertz can assume that the whole compression takes place at the point of impact, the remainder of the sphere acting as a rigid body. The problem of two bars each of length $2r$ does not differ widely from the Hertz problem of two balls of radius r. If $P/Q=3/2$,

$$P_0=\pi r^2 k(\alpha/l)^\beta=1{\cdot}06r^{1\cdot2}k\alpha^{3\cdot2}.$$

For the problem of two balls, Hertz gives

$$P_0={\cdot}53r^{1/2}k\alpha^{3\cdot2}.$$

It follows from the foregoing that in the case of two bodies whose lateral dimensions are comparable with the longitudinal ones, the stress is dispersed through the bodies and the energy in consequence is scattered. The presence of a cylindrical boundary introduces the condition that the stress has no normal component at the boundary; the lines of pressure tend to become parallel to the boundary, the stream of energy being guided along the bars. The extent to which this can be accomplished depends on the ease or otherwise with which radial strain can be transformed into longitudinal strain.

Now, $$\frac{\text{Longitudinal energy at boundary}}{\text{Radial energy at boundary}}=\frac{2\pi^2 m\nu^2 b^2}{2\pi^2 m\nu^2 a^2},$$

where m is the mass of the vibrator, ν its frequency, b the longitudinal amplitude, and a the corresponding radial amplitude. If l is the length of the bar and r its radius,

$$b/a=dl/dr=\sigma l/r, \text{ where } \sigma \text{ is Poisson's ratio.}$$

The ease or otherwise with which a divergent wave of compression is transformed into a plane longitudinal wave therefore depends on the quantity $\sigma l/r$ occurring as an index in the expression

$$(P_0/Ak)^{Q+\kappa l/r}=(\alpha/l)^{P+\kappa l/r}.$$

K should thus be proportional to Poisson's ratio.

The case of loaded bars follows from this point of view. Let $m_0=\pi r^2 l\rho$ be the mass of the bar and $M=\pi r^2 l_1\rho$ the mass of the collar. Then

$$2P_0=-(M+m_0)\frac{d^2\alpha}{dt^2},$$

where P_0 is the pressure between the bodies and α is the

relative movement of their centres of gravity. The pressure P_0 can also be written

$$P_0 = \pi r^2 k(\alpha'/l)^\beta,$$

where α' is a quantity defined by the equation $(m_0 + M)\alpha' = m_0\alpha$. Substituting for α',

$$P_0 = \pi r^2 k(m_0\alpha/\overline{M + m_0}l)^\beta.$$

The law is not obeyed for large initial velocities of approach, as the collar is not able to take up the full movement. The time of impact consequently shows a rapid falling off for large values of v as indicated in a previous paper.

Alternatively, it has been shown by experiment that in the case of equal bars of radius r, length l, and density ρ, each loaded with a mass M, the relationship between the time of collision and the relative velocity of approach is of the form $t = Av^\gamma$ where $-1/\gamma = A'' + K(l + l_1)/r$, l_1 being given by the equation $\pi r^2 l_1 \rho = M$. This generalization of the equation $-1/\gamma = A'' + Kl/r$ which applies to simple bars, must follow since the motion of the load is longitudinal. The quantity $\sigma l/r$ is increased to $\sigma(\pi r^2 l\rho + M)/\pi r^3\rho = \sigma(l + l_1)/r$. This suggests that the case of loaded bars is met if $l + l_1$ replaces l in all the formulæ. Thus

$$2P_0 = -(M + m_0)\frac{d^2\alpha}{dt^2},$$

and

$$P_0 = \pi r^2 k(\alpha/l + l_1)^\beta,$$

which give a time of collision

$$t = \frac{2v^{1-\beta}}{4^{1/1+\beta}}(\beta+1)^{1/1+\beta}(\rho/k)^{1/1+\beta}(l + l_1)\int_0^1 \frac{dx}{(1 - x^{\beta+1})^{1/2}},$$

i. e. the bars have the same time of impact as two bars of the same diameter and material and of such a length L that $\pi r^2 L\rho = M + m_0$. This has been found to be the case.

The suggestion, from these experiments, that the time of impact depends fundamentally on the mass and not on the length of the bars leads to a formula for t of the form $t = Cl^{1/3}r^{2/3}v^\gamma(\rho/E)^{\gamma+1/2}$. The possibilities of this formula have been investigated but it has proved unsatisfactory.

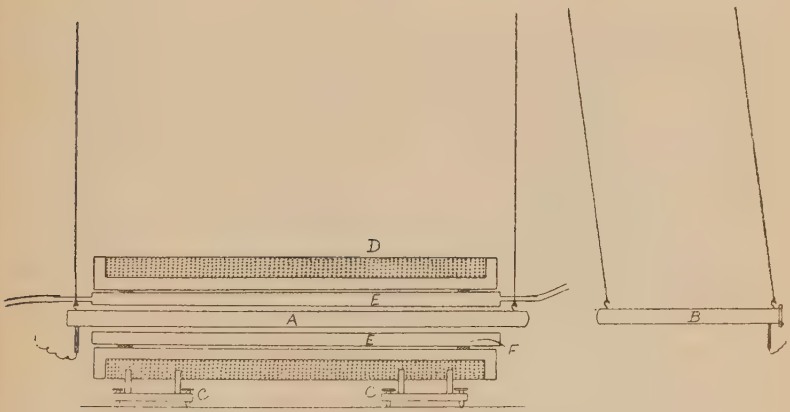
The Effect of a Longitudinal Magnetic Field on the Time of Impact of Steel Bars.

It seems possible that under the action of a magnetic field the time of impact may be slightly modified since the magnetic properties of iron are molecular in origin.

The arrangement used in the experiment is shown in fig. 2. A steel bar A (61·25 cm. long) is suspended so as to

lie symmetrical with respect to and along the axis of a solenoid D wound round an ebonite cylinder. The solenoid rests on small V-shaped blocks C fitted with levelling screws for adjusting the height. Between the solenoid and the bar is fitted a water-jacket through which a stream of water runs continuously. This prevents any heating of the bar due to conduction. In the light of experience the precaution now seems almost unnecessary, but at the time it was considered desirable to eliminate all spurious effects as completely as possible. E and D are separated by two bands of insulating tape wound round the jacket, and an annular space of $\frac{1}{4}$ in. separates the jacket and the bar. The solenoid

Fig 2.



needs accurate adjustment. When a current is passed round the solenoid, the bar is in general moved bodily so as to embrace the greatest number of lines of the magnetizing field. This motion can, however, be removed completely by a suitable disposition of the coil. To test this, the rounded end of the bar is viewed broadside on with a microscope and the solenoid is moved about until the end of the bar coincides with the same scale division whether the current is on or off. Any sideways movement is similarly eliminated. In order to get rid of the strong magnetic attraction between A and B which would ordinarily render the experiment impossible (as successive impacts would follow one another so rapidly that the condenser would be discharged completely), B is made of Hadfield's non-magnetic steel.

An ammeter and reversing commutator are arranged in the circuit feeding the solenoid, and to avoid hysteresis

heating the current is obtained from accumulators. Several long sets of observations have been taken, of which a typical one is quoted below.

TABLE I.

Velocity of bar B at impact = 10·352 cm./sec.

Magnetizing field : gauss.	Galvanometer throw for successive impacts.	Mean galv. throw corrected for damping.
0	130, 130·2, 128·5, 129, 129, 130, 129	130·7
+415	129·5, 128·5, 129·5, 129·3, 129·5, 129, 127, 129·5, 129·5, 129·3, 128, 129·3	130·45
	Between these sets of readings the bars are not allowed to collide.	
−415	130·4, *120·0*, 130·2, 130·2, 129·7, 129·2, 129·2, 129·7, 129·7, 129·2, 129·2, 129·2	130·4
	Between these sets of readings the bars are not allowed to collide.	
0	128, *126·2*, 129·2, *127·2*, 128·2, 128·2, *125·7* 129·2, 129·2, *126·7*, *126·2*, 129·7, 129·9, 129·2, 129·2, 130, *125·7*, 128·2, *126·2*, 129·2, 128·7, 129, 129·2, 129·2, *125·2*, 128·7, 128·7, 128·7, 129·2, 129·2, *125·7*, 129·5, 129·5, 128, 129·5, 129·5, 130·3, 130·2, 127·7, 129·7, 129·7, 128·5, 128·2, 129·7, 129·7, 129·7, 129·2, 128, 128·2, 128·2, *123·7*, 129·2, 129·2, 130, 129·7, 129·7, 129·7, 129·7, 129·2, 129·2, 129·4.	130·5
	An interval of twelve hours occurs here, but the surfaces of the bars are not touched before proceeding with the following.	
0	128·5, 128·5, 130, 129, 129·3, 129, 129·5, 129·5, 129·5, 129, 129, 129·5, 129, 129, 129·5, 129. 129·3, 128·5, 129, 129·5, 129, 129, 129, 129·3, *128*, 129·5.	130·5
	The bar A is now completely demagnetized and a long set of readings is taken to test their constancy.	
0	132, 131·8, 131·8, 131·0, 131·0, 132·0, 131·0, 131·5, 131, 131·5, 131, 131·5, 131·2, 132, 131, 131, 131, 131·5, 131, 131, 131, 131, 131·5, 130·5, 130·5, 130·5, 131, 131, 130·5, 130·5, 131, 130·8, 130·5, 130·5, 130·5, 130·3, 131, 130·5, 130·5, 130·5, 131, 130·8, 130·8, 130·3, 130·3, 130·8, 130·8, 130·5, 130·8, 130, 130·3, 131, 130·5, 130·8, 130·3, 131, 130·8, 130·5, 130·8, 130·8, 130·5, 130·8, 130·3, 130·3, 130·5, 130·5, 131, 130·8, 131, 130·5.	132·3

The above readings, which were taken with extreme care, show that the average values for the magnetized state are the same as for the demagnetized state. They do, however, exhibit quite an interesting effect. On examining sets 1 and 2 of the above table, it is seen that the individual readings vary very little about the mean. On the other hand, the second impact of set 3 (corresponding to the reversal of the magnetizing field together with the delivery of one previous blow to the bar) is abnormally short, after which they are steady again.

In set 4, taken immediately after switching off the magnetizing field, the 2nd, 4th, 7th, 10th, 11th, 17th, 19th, 26th, 32nd, and 52nd impacts are short, after which they are comparatively steady.

It is suggested that the erratic readings do actually reflect a change in the state of the metal due to the blow delivered at the end. Changes of the order observed can be produced by the slight stretching of a suspension, a slight departure from perfect alignment, or the presence of a speck of dust at the place of impact. The impacts immediately preceding or following these abnormal ones are, however, perfectly normal usually, which would not be the case if the effects were due to any of the alternative explanations given above. In the latter case a permanent change of reading would result.

In order to pursue the inquiry whether the abnormal readings can be taken to indicate an internal change as opposed to a variation in external conditions, several extended series of observations were taken with the same bars after demagnetization, and for zero magnetizing field. One series of seventy consecutive readings is given in Table I. It is seen that there is not a single abnormal impact. There is a tendency for the impacts to get progressively shorter, but this can be traced to a change in the external conditions, viz. a gradual flattening of the end of the non-magnetic steel bar.

The abnormal impacts are therefore probably due to a change in the magnetic state of the bar produced by the blow on the end. It has previously been shown that the duration of impact is related to the initial velocity of approach by the equation $t = Av^{\gamma}$. The kinetic energy of the impinging bar at the instant of impact $= \frac{1}{2}m_0v^2$. If it is legitimate to assume that this energy, during impact, represents the energy of the waves of compression passing through the bars

$$t = \text{constant}/(\text{Energy in compression wave})^{-\gamma/2}.$$

Any change in the energy will be reflected in the duration of the impact. The short impacts may indicate that the

wave while passing through the bar has become enriched in energy due to the breaking up of aggregates peculiar to the magnetized state into aggregates corresponding to the normal state.

The Effect of Temperature on the Time of Impact of Steel Bars.

This short series of experiments sets out to determine the effect of temperature on the duration of impact. From the small effects obtained, it is clear that the results of the experiments on the effects of a magnetizing field, described in the last section, are in no way associated with temperature variations in the bars.

For this investigation each bar is provided with a steam-jacket from which it is insulated by thin ebonite plugs, and the times of impact are found at 18° C. and 100° C. The results are given below.

TABLE II.

Temperature of bar.	Resistance in circuit: ohms.	Galvanometer throw.	Mean value of throw.
18° C.	187	189, 189·5, 189·5, 189, 189·5, 189, 189·5, 189·5, 189, 189·5	189·3
	287	154·2, 154, 154·2, 153·5, 153·8, 154·5, 154, 154	154·03
	387	130·5, 130·2, 130, 130, 131, 131, 131, 130·5, 131·5, 131	130·67
	687	90, 90, 90, 90, 90, 89·5	89·92
99·5° C.	187	189·5, 189·5, 189, 189·5, 189·5, 189·5, 189·5	189·43
	287	154·5, 154·5, 154·5, 154·2, 154·5 154·7, 154 7	154·52
	387	131, 130, 131, 131, 130·3	130·66
	687	90, 90, 90, 90, 89·5	89·92

The readings at 18° C. seem to be slightly less than those at 100° C., indicating that the duration of impact is slightly less at the lower temperature, but on the whole the change is quite negligible.

In conclusion, I wish to thank Mr. C. Barnes, M.Sc., for his assistance in the experimental portions of the paper, and to thank also the Grants' Committee of the Royal Society for defraying the cost of the apparatus.

University of Leeds.
August 31st, 1923.

XVIII. *The Emission Spectra of Mixed Alkali Vapours.*

By F. H. Newman, *D.Sc., A.R.C.S., Professor of Physics, University College, Exeter* *.

(Plate IV.)

1. *Introduction.*

In a previous paper † the author has described an emission continuous spectrum which is obtained when an electric arc is passed between two pools of the alloy of sodium and potassium. The spectrum consists of two parts—one in the red region and the second in the blue-green region, the former being by far the more intense. Another interesting feature was the formation of an emission fluting spectrum, the heads of the bands being situated between the limits λ 6700 and λ 5500. These results have been studied in greater detail in the present experiments.

The absorption by sodium vapour is of two types, a principal series which is characterized by the empirical formula of the emission principal series of which the D lines form the first member. In addition, there are a number of very complicated channelled absorption bands which do not seem to influence dispersion. One of these bands lies in the red, λ 5800 to the extreme limit of the visible red, and the other in the blue-green region, λ 5700–λ 4500, though with dense vapour they practically meet in the yellow. Wood ‡ has found that there are about 6000 absorption lines in these bands in the visible region, and the regions transmitted between the absorption lines are as narrow apparently as the emission lines of the iron arc. There are on an average 60–70 absorption lines within a space of only 12 Ångström units in width. These lines, taken collectively, form themselves into a number of groups which resemble the groups seen in the absorption spectra of iodine and bromine, and in certain banded emission spectra. Although there are no actual emission spectra corresponding to the complicated absorption ones, Wood has shown that these bands are connected with the fluorescent spectra; and probably when sodium vapour is illuminated with a powerful beam of white light, its fluorescent spectrum is the exact complement of the absorption spectrum. If the vapour is stimulated with monochromatic light, a series of bright lines spaced at nearly equal intervals along a normal spectrum and separated by a distance

* Communicated by the Author.

† Phil. Mag. xlvi. (September 1923).

‡ 'Physical Optics,' 1911.

equal to 37 Ångström units is obtained. Various series of lines can be made prominent by altering the wave-length of the exciting light. Stimulation by λ 4800 from the cadmium arc produces a series of bright lines spaced at nearly equal intervals along a normal spectrum. The bismuth arc produces a resonance spectrum consisting of a series in the blue which with long exposures can be traced well up into the green, and a large number of lines in the yellow-green part of the spectrum. With the lithium arc, definite series of lines can be excited in the red region, and the four exciting lines all produce different lines in the resonance spectrum.

The existence of bands with sodium vapour is due to associated atoms of the alkali vapour. The pure rotation band-spectra, as observed by Rubens* with water vapour, can be explained by the change in the rotational quantum number by ± 1. Bands in the visible region occur when there is a change in the configuration of the molecule which results in a decrease in the internal energy at the same time as the change in angular momentum. This change of configuration of molecule consists in a re-arrangement of the electrons or the atoms within the molecule.

If the dispersion of the instrument is not sufficient, some band-spectra appear continuous, especially those which consist of the so-called many-line spectra. In addition, there are really continuous spectra both in emission and absorption. This continuous character is especially marked in the case of the absorption bands observed with sodium vapour. There is a selective absorption in a continuous spectrum region beginning at the limit of the principal series and extending into the ultra-violet. This confirms the assumption that the absorption of energy corresponding to lines in the principal series of sodium results in fixed states of the atom in which one of the electrons revolves in larger and larger orbits. This continuous absorption corresponds to transitions from the normal state to states in which the electron is in a position to remove itself infinitely far from the nucleus. A continuous spectrum arises if an originally free electron is taken up into the configuration of an originally ionized atom and the latter is neutralized.

In the case of iodine, Steubing† has photographed an emission spectrum of iodine which appears as a continuous band, even at dispersions that completely resolve the very narrow band-spectrum of iodine. The atom of the latter tends to perfect the 8-shell arrangement by taking a free

* *Berliner Ber.* 1913, p. 513.

† *Zeitschrift für Phys.* v. (1921).

electron, and as a result it is highly electronegative in properties. Hence, it need not be ionized to attract an electron to itself, but the electron affinity suffices to effect the taking up of the free electron.

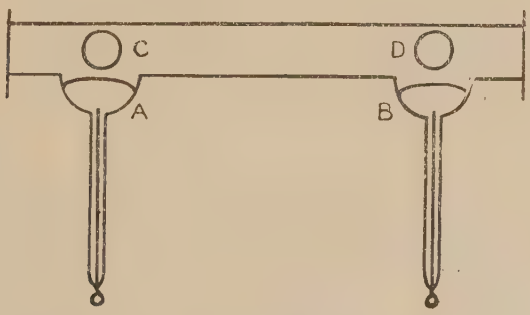
Wood found that potassium vapour has a channelled absorption spectrum consisting of more or less regularly-spaced bands lying within the orange-red region limited by λ 6346 and λ 6667, and McLennan and Ainslie * have shown that, in addition to these bands, there is a series of fine channelling beyond λ 7699 towards the infra-red, and bands in the neighbourhood of λ 4273 when vapour of high density is used. Vapours of the other alkali metals also exhibit channelled absorption bands.

It is to be expected that when we are dealing with mixed vapours of more than one alkali there will be bands due to each element, and in addition bands due to the association of atoms from the different elements.

2. *Experimental Results.*

The form of quartz tube used in the present work is shown in fig. 1. Sodium potassium alloy—equal parts by weight of

Fig. 1.



sodium and potassium—formed the two electrodes at A and B. The ends of the tube were closed with glass plates cemented with wax, and C and D were two side-tubes, so that the spectra immediately at the anode and cathode could be observed, and photographed on panchromatic plates with a constant deviation type of spectrometer. The arc could be readily maintained with an applied potential difference of 50 volts, and the current was controlled by a variable resistance in series with the tube. Sometimes the arc was started by warming the tube with a bunsen flame ; otherwise

* Proc Roy. Soc. ciii. (1923).

it was necessary to apply a discharge from an induction coil in addition. Once started, the lamp required no further attention, there being no gases liberated. The heat produced by the current was sufficient to produce the requisite vapour-pressure, but for high vapour-pressures a coil of nichrome wire wound round the tube was heated electrically.

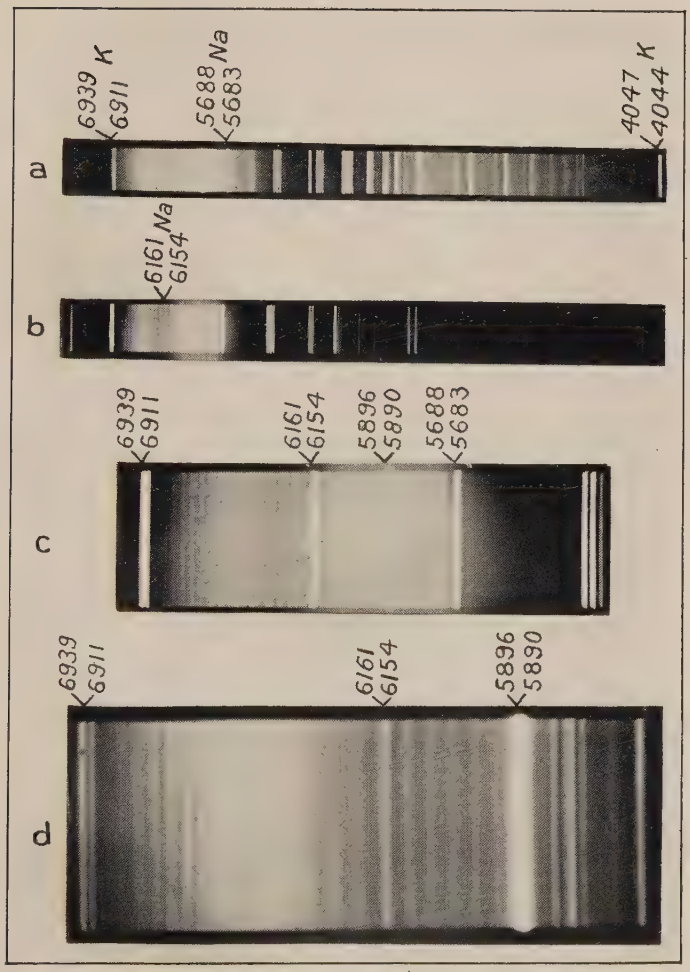
In addition to the well-known series of sodium and potassium lines, there were exhibited the bands previously observed. With small vapour-pressures they were faint, increasing in brightness as the tube was heated or the current increased. The increased intensity is due to the greater vapour-pressure. As the latter was raised, the bands merged into a continuous band which extended from λ 6780 to λ 5485. In addition, at higher temperatures another band which appeared continuous between the limits λ 4750, λ 4040 was exhibited. This latter band was noted when the alloy was heated to 300° C., although it is difficult to estimate the actual temperature within the tube. It is certain that this temperature exceeds that registered by a thermometer placed within the electric heater. The admission of a little nitrogen or hydrogen to the lamp caused the appearance of the band in the red region, suggesting that the total pressure within the tube decides the formation of this band.

The instrument used was not of sufficient dispersive power to examine the structure of the individual bands constituting the channelling in the red, but they were slightly degraded towards the blue. These bands extended from λ 6776 to λ 5931, but from λ 5931 to λ 5485 merged into a continuous band which could not be resolved. Unfortunately, the intense radiation emitted by the D lines of sodium prevented observation of the wave-lengths in the immediate neighbourhood of λ 5893 (Pl. IV.).

The series of bands in the red region which formed a channelled spectrum were measured, and gave the mean wave-lengths recorded in Table I.

The continuous band in the green-blue probably consists of a large number of bands. At low vapour-pressures they are too faint to be observed, and merge into the continuous band at higher pressures. This seems to show that, with the vapour at moderate density, only a small proportion of the vapour is in the molecular state suitable for giving this particular type of emission.

It is probable that the complete banded spectrum arises from three distinct sets of molecules—viz., potassium, sodium, and molecules consisting of associated atoms of both elements. The sodium and potassium arcs are difficult to regulate, and



Emission Banded Spectra of the Vapour from Sodium-Potassium Alloy.

it was impossible to detect banded emission-spectra from these elements separately.

TABLE I.

λ A.U.	ν = Frequency.	$\delta\nu$.
5931	16861	
		130
5977	16731	
		111
6017	16620	
		124
6062	16496	
		127
6109	16369	
6161 Na line	16231	
6154 Na line	16250	
6193	16147	
		101
6232	16046	
		130
6283	15916	
		118
6330 †	15798	
		104
6372 †	15694	
		121
6420 †	15573	
		98
6462	15475	
		97
6503	15378	
		80
6537	15298	
		93
6577	15205	
		104
6622	15101	
		108
6670	14993	
		116
6722	14877	
		119
6776	14758	

† Sharp.

3. *Description of Plate IV.*

(*a*) Continuous band from λ 6780 to λ 5485, and from λ 4750 to λ 4040.

(*b*) Channelling consisting of bands from λ 6776 to λ 5931, and a continuous band from λ 5931 to λ 5485.

(*c*) An enlarged spectrogram of a part of (*b*), showing the bands in the red region.

(*d*) Bands in the red region, the vapour-pressure being greater than in (*c*).

XIX. *The Motion of a Rigid Solid according to the Relativity Principle.* *By* G. Lemaître, *D. ès Sc.* (*Louvain*) *.

[Since previous work on M. Lemaître's problem — intermediate between the special and general relativity theories—is probably not very familiar to English readers, a brief introduction may be desirable. Simultaneity is relative: but relative to what? In the special theory it is relative to a uniformly moving particle, it being assumed that we have a means of fixing constant orientation in space—or, to combine the two conditions, relative to a "Galilean" frame of axes. If we seek to materialise the Galilean frame, we must substitute a rigid solid. Now an actual solid of reference may differ from a Galilean frame, either because it is rotating or because its motion as a whole is accelerated (by non-gravitational forces). This invites a generalization of the definition of simultaneity, so that we may consider simultaneity relative to a generally moving solid. We still limit ourselves to a particular case of the general theory, because our four-dimensional continuum of space-time is assumed to be flat (Euclidean); but, as M. Lemaître shows, in certain cases the resolution into space and time separately gives a three-dimensional space which is curved. By studying this transition we see how the notion of simultaneity gradually fades out in passing from the special to the general theory: in motions of class A, it remains definite but is no longer a matter of permanent adjustment of clocks; in motions of class B, it exists only between neighbouring points and becomes ambiguous for distant points unless a route of connexion is specified. The definition of a rigid body appeals to space-measures: but simultaneity is involved in the separation of space-measures from time-measures; and simultaneity in its turn is relative to the rigid body. It is of interest to note how this logical tangle is approached.—A. S. Eddington.]

1. THE variable motion of a solid and the correlative aspects of phenomena for moving observers is an intermediate problem between the first relativity theory of uniform motion and the general principle by which gravitation is accounted for.

The fundamental paper on this question is that of M. Born †, who has fixed the general meaning of the interval and its spatio-temporal interpretation as retained in the general theory. The necessity of throwing over the Euclidean Geometry in Born's conception has been pointed out by Ehrenfest ‡. Finally, G. Herglotz § has investigated

* Communicated by Prof. A. S. Eddington, F.R.S.
† *Ann. d. Phys.* xxx. p. 1 (1909).
‡ *Phys. Zeitschr.* x. p. 918 (1909).
§ *Ann. d. Phys.* xxxi. p. 393 (1910).

the most general possible movement of a solid in a Galilean field. We will summarize these results and deduce developments and applications suggested by the present point of view of the Relativity Theory.

All physical notion of space and time may be expressed by measures of *interval* or generalized length of lines in the four-dimensional Universe. The time of a chronometer (proper time) is the interval along its line of universe.

Simultaneity as defined by light-signals observed on a definite body is such that the direction of a line of simultaneity is conjugate to the direction of the lines of universe described by the body with respect to the light-cone, or locus of vanishing intervals, passing through the point.

In the geometrical interpretation, lines of simultaneity are orthogonal trajectories of the lines of universe described by the body.

A spatial length on a body is the interval along its lines of simultaneity.

In the particular case ordinarily considered there is a definite set of point-events which can be joined to a given point-event by lines of simultaneity. In this case it is possible to assign to this set of points a number which is called the *time of simultaneity* on the body. All the point-events of same time of simultaneity, and only these, can be joined by lines of simultaneity. The definition of simultaneity does not depend on the path by which it is defined ; its mathematical expression is an exact differential, and it may therefore be used as a coordinate.

But in the general case it is possible to draw a line of simultaneity through any two arbitrary point-events, and there is no possible definition of a time of simultaneity. Examples of this will be given below.

Now a rigid solid is a body on which all the relative spatial distances of points, measured on the body, are left unaltered by the movement. It must be observed that the definition of spatial distance refers only to the line of simultaneity and is independent of the existence of a time of simultaneity.

The possible movements of a solid have been divided by Herglotz into two classes according to the existence of a time of simultaneity or not. The movements of class A, *i. e.* with a time of simultaneity, are completely determined by the arbitrary motion of one point.

In this case the interval can be written, by a convenient

choice of coordinates, in the form

$$ds^2 = -dx^2 - dy^2 - dz^2 + \phi^2 dt^2, \quad . \quad . \quad . \quad . \quad (1)$$

where ϕ is a function of x, y, z, t.

In the movements of class B, the motion of every point must be expressed by use of a parameter which is not a time of simultaneity. This parameter may be chosen so that the interval between two arbitrary points is not altered when the same number is added to their respective parameters. The line of universe described by every point is transformed into itself by the movement, and the general movement of class B is a one-parameter group of movements in the Universe.

On the basis of these results of Herglotz we begin by investigating movements of class A for which the given motion of a point is rectilinear.

2. *Rectilinear movement.*

Let v be an arbitrary function of t whose derivative is g. Then

$$\left.\begin{aligned} d\xi &= \cosh v\, dx + (1+gx)\sinh v\, dt, \quad d\eta = dy, \quad d\zeta = dz, \\ d\tau &= \sinh v\, dx + (1+gx)\cosh v\, dt, \end{aligned}\right\} \quad (2)$$

are the exact differentials of the functions

$$\left.\begin{aligned} \xi &= x\cosh v + \int \sinh v\, dt, \quad \eta = y, \quad \zeta = z, \\ \tau &= x\sinh v + \int \cosh v\, dt, \end{aligned}\right\} \quad . \quad . \quad (3)$$

which define a transformation from the coordinates ξ, η, ζ, τ to x, y, z, t.

The interval in a Galilean field

$$ds^2 = -d\xi^2 - d\eta^2 - d\zeta^2 + d\tau^2 \quad . \quad . \quad . \quad . \quad (4)$$

is transformed into

$$ds^2 = -dx^2 - dy^2 - dz^2 + (1+gx)^2\, dt^2. \quad . \quad (5)$$

For given values of x, y, z the equations (3) define an arbitrary movement of a point of the solid, parallel to the ξ axis. Equation (5) is thus the general expression of rectilinear movements of class A.

It is easy to see that t is truly a time of simultaneity. The directions for which $dt = 0$ are conjugate to the direction

$dx=dy=dz=0$ of the lines of universe of the moving points with regard to the light-cone $ds=0$.

The spatial distance is

$$d\sigma^2=dx^2+dy^2+dz^2,$$

obtained by putting $dt=0$ in the expression of ds^2.

The time shown by the chronometers of the moving system (proper time) is

$$\int_{t_1}^{t_2}(1+gx)\,dt=t_2-t_1+x(v_2-v_1). \quad . \quad . \quad . \quad . \quad (6)$$

Measures of simultaneity (by light-signals) will show a variable difference between the simultaneous readings of the chronometers. The difference between the time measured by the chronometers from the epoch t_1 to the epoch t_2 at two points of coordinates x and $x+\Delta x$ is

$$(v_2-v_1)\Delta x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which depends on the values of v at the epochs only.

v is connected with the velocity V of the system by the relation

$$\mathrm{V}=\frac{d\xi}{d\tau}=\tanh v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is obtained by putting $dx=0$ in (2).

Conversely,

$$v=\tfrac{1}{2}\log\frac{1+\mathrm{V}}{1-\mathrm{V}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The velocity does not depend on x. That means that every point has the same speed at the same time t: i. e., simultaneously for the moving system. For the fixed system, on the contrary (time τ), the speed is taken up progressively so that the length may be altered according to the Lorentz contraction, variable with the variable velocity.

An interesting application may be made to the problem of the change of velocity of a Galilean observer. Suppose that before the time t_1 the velocity has a constant value V_1 corresponding to v_1, and then varies in amount but keeping the same direction, and finally remains constant after the time t_2.

Since the system is a Galilean one, before the epoch t_1

the chronometers are able to show the time of simultaneity. During the change of speed (t_1, t_2) the simultaneity will be disturbed and, after t_2, the chronometers will show a constant difference between their simultaneous readings.

This difference is given by (7) and is

$$(v_2-v_1)\Delta x = \frac{\Delta x}{2}\log\frac{1+V_2}{1-V_2}\cdot\frac{1-V_1}{1+V_1}. \quad . \quad . \quad (10)$$

Δx is the distance measured in the sense of the movement. V_1 and V_2 are the starting and final velocities, the velocity of light being taken as unit. The law of velocity variation is immaterial.

3. *Movement with constant acceleration.*

When g is a constant, it is possible to carry out the integrations in the transformation (3). Then it may be written :

$$\left.\begin{aligned} 1+g\xi &= (1+gx)\cosh gt, \\ g\tau &= (1+gx)\sinh gt. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

By elimination of t_1 we obtain the equations of the movement

$$(1+g\xi)^2-(g\tau)^2 = (1+gx)^2, \quad . \quad . \quad . \quad . \quad (12)$$

or, changing the origin,

$$\xi^2-\tau^2 = x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is Born's hyperbolic movement. The moving points describe concentric hyperbolas in the plane $\xi\tau$.

The interval (5) may be simplified by using new coordinates *

$$ds^2 = -dx^2-dy^2-dz^2+x^2dt^2. \quad . \quad . \quad . \quad . \quad (14)$$

The acceleration of a free point, or geodesic line of the universe, is

$$\frac{d^2x}{ds^2} = -x\left(\frac{dt}{ds}\right)^2, \quad \frac{d^2y}{ds^2} = 0, \quad \frac{d^2z}{ds^2} = 0. \quad . \quad . \quad . \quad (15)$$

The trajectory is a plane and we have for convenient choice of the coordinates

$$ds = \lambda\, dy, \quad dz = 0.$$

* This formula is given by Pauli, 'Relativitätstheorie,' p. 648.

Then from (14)

$$x^2\left(\frac{dt}{dy}\right)^2 = 1+\lambda^2+\left(\frac{dx}{dy}\right)^2, \quad . \quad . \quad . \quad . \quad (16)$$

and from (15)

$$x\frac{d^2x}{dy^2}+\left(\frac{dx}{dy}\right)^2+1+\lambda^2 = 0$$

or

$$\frac{d}{dy}\left(x\frac{dx}{dy}\right)+1+\lambda^2 = 0.$$

The solution may be written (a, b, c constant):

$$\left.\begin{aligned} x &= a\cos\theta, \\ y &= b\sin\theta+c, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

with

$$\frac{a^2}{b^2} = 1+\lambda^2.$$

The trajectories are ellipses with minor axis in the plane $x=0$. The time t is given from (16) by

$$t-t_0 = \int_{\theta_0}^{\theta}\frac{d\theta}{\cos\theta} = \log\frac{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)}. \quad . \quad . \quad (18)$$

When $\lambda=0$, $ds=0$, the trajectories are rays of light. They are circles with centre on $x=0$.

For comparison with the Schwarzschild-Einstein formula for the field of a particle, it is convenient to effect the transformation

$$(1+gx)^2 = 1+2g\mathrm{X}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Then the equation of the field may be written (g constant):

$$ds^2 = -\frac{d\mathrm{X}^2}{1+2g\mathrm{X}}-dy^2-dz^2+(1+2g\mathrm{X})\,dt^2. \quad . \quad (20)$$

4. *General movement of Class* A.

It is easy to find the general expression of the function ϕ of x, y, z, t in Herglotz's equation of a field of class A:

$$ds^2 = -dx^2-dy^2-dz^2+\phi^2dt^2. \quad . \quad . \quad . \quad (21).$$

The necessary condition that this expression is reducible by a change of coordinates to the Galilean form is that the

Riemann tensor vanishes. Its components which do not vanish identically are :

$$B_{ik44}=B_{44ik}=-B_{ik4k}=-B_{4i4k}=\phi\frac{\partial^2\phi}{\partial x_i\partial x_k},\quad (i,k=1,2,3), \qquad (22)$$

where x_1, x_2, x_3 are written for x, y, z.

Derivatives with regard to t do not occur.

ϕ must be of the form :

$$\phi = lx+my+nz+p,$$

where l, m, n, p are functions of t and the field of a moving system of class A is of the form

$$ds^2=-dx^2-dy^2-dz^2+(lx+my+nz+p)^2dt^2. \qquad (23)$$

p may be taken equal to one by a change of the coordinate t.

The transformation may be written as follows :

$$\left.\begin{aligned}\xi &= a_{11}x+a_{12}y+a_{13}+\int a_{14}dt,\\ \eta &= a_{21}x+a_{22}y+a_{23}+\int a_{24}dt,\\ \zeta &= a_{31}x+a_{32}y+a_{33}+\int a_{34}dt,\\ \tau &= a_{41}x+a_{42}y+a_{43}+\int a_{44}dt,\end{aligned}\right\} \qquad (24)$$

where the 16 coefficients are functions of t defined by the 16 differential equations :

$$\left.\begin{aligned}\frac{a'_{11}}{a_{14}}=\frac{a'_{21}}{a_{24}}=\frac{a'_{31}}{a_{34}}=\frac{a'_{41}}{a_{44}}&=l\equiv l_1,\\ \frac{a'_{12}}{a_{14}}=\frac{a'_{22}}{a_{24}}=\frac{a'_{32}}{a_{34}}=\frac{a'_{42}}{a_{44}}&=m\equiv l_2,\\ \frac{a'_{13}}{a_{14}}=\frac{a'_{23}}{a_{24}}=\frac{a'_{33}}{a_{34}}=\frac{a'_{43}}{a_{44}}&=n\equiv l_3,\\ a'_{1\sigma}a_{14}+a'_{2\sigma}a_{24}+a'_{3\sigma}a_{34}-a'_{4\sigma}a_{44}&\\ +a_{1\sigma}a'_{14}+a_{2\sigma}a'_{24}+a_{3\sigma}a'_{34}-a_{4\sigma}a'_{44}&=0,\\ (\sigma=1,2,3,4)&\end{aligned}\right\} \qquad (25)$$

with initial values satisfying

$$\left.\begin{aligned}a_{1\sigma}^2+a_{2\sigma}^2+a_{3\sigma}^2-a_{4\sigma}^2&=\begin{cases}1 \text{ for } \sigma=1,2,3,\\ -1 \text{ for } \sigma=4,\end{cases}\\ a_{1\sigma}a_{1\tau}+a_{2\sigma}a_{2\tau}+a_{3\sigma}a_{3\tau}-a_{4\sigma}a_{4\tau}&=0\\ (\sigma\neq\tau=1,2,3,4).&\end{aligned}\right\} \qquad (26)$$

These initial values are possible as they correspond to a Lorentz transformation.

Let us differentiate these equations (26), and observe that for σ or $\tau=4$ the differential is null by (25) and for $\sigma, \tau \neq 4$

$$a_{1\sigma}a'_{1\tau}+a_{2\sigma}a'_{2\tau}+a_{3\sigma}a'_{3\tau}-a_{4\sigma}a'_{4\tau}$$
$$=l_\tau(a_{1\sigma}a_{14}+a_{2\sigma}a_{24}+a_{3\sigma}a_{34}-a_{4\sigma}a_{44})=0.$$

We see that the initial conditions (26) are verified everywhere.

Then

$$d\xi=a_{11}\,dx+a_{12}\,dy+a_{13}\,dz+(a'_{11}x+a'_{12}y+a'_{13}z+a_{14})\,dt,$$

and from the above conditions

$$-d\xi^2-d\eta^2-d\zeta^2+d\tau^2$$
$$=-dx^2-dy^2-dz^2+(lx+my+nz+1)^2dt^2. \quad (23\text{ bis})$$

Observe that from (26)

$$a_{\sigma\tau}=a_{\tau\sigma}.$$

We will now verify that the movement of one point may be taken arbitrarily. For $x=y=z=0$, we have $ds=dt$ and the velocity of the point is

$$\frac{d\xi}{ds}=a_{14},\quad \frac{d\eta}{ds}=a_{24},\quad \frac{d\zeta}{ds}=a_{34},\quad \frac{d\tau}{ds}=a_{44},$$

with

$$a_{14}^2+a_{24}^2+a_{34}^2-a_{41}^2=-1.$$

The functions l, m, n of $t=s$,

$$l=\frac{1}{\frac{d\tau}{ds}}\frac{d^2\xi}{ds^2},\quad m=\frac{1}{\frac{d\tau}{ds}}\frac{d^2\eta}{ds^2},\quad n=\frac{1}{\frac{d\tau}{ds}}\frac{d^2\zeta}{ds^2}, \quad . \quad . \quad (27)$$

may be determined for an arbitrary motion of the point $x=y=z=0$.

For every point we have

$$u=\frac{d\xi}{d\tau}=\frac{a_{14}}{a_{44}},\quad v=\frac{d\eta}{d\tau}=\frac{a_{24}}{a_{44}},\quad w=\frac{d\zeta}{d\tau}=\frac{a_{34}}{a_{44}}, \quad . \quad (28)$$

with

$$a_{44}=(1-u^2-v^2-w^2)^{-\frac{1}{2}}=(1-V^2)^{-\frac{1}{2}}. \quad . \quad (29)$$

Change of velocity for a non-rectilinear movement.

The proper time shown by chronometers of the moving system between two epochs t_1 and t_2 is

$$\int_{t_1}^{t_2}(lx+my+nz+1)\,dt = t_2-t_1+\int\frac{1}{a_{44}}(a'_{41}x+a'_{42}y+a'_{43}z)\,dt. \quad . \quad . \quad . \quad (30)$$

The difference of the time measured by two chronometers between the two epochs generally depends on the particular law of velocity variation. We have seen that the difference is independent of this law when the velocity keeps the same direction. The same thing happens when a_{44} is a constant, *i. e.* when V is a constant. Then the change is

$$\frac{1}{\sqrt{1-V^2}}[(u_2-u_1)\Delta x+(v_2-v_1)\Delta y+(w_2-w_1)\Delta z]. \quad (31)$$

The difference of the simultaneous readings of two chronometers after a change of direction with constant speed does not depend on the described curve but is proportional to the distance of the chronometers and to the difference of the projections of the starting and final velocities on the line which joins them.

Acceleration of free points.

Equations of the geodesics are

$$\frac{d^2x}{ds^2} = -l(lx+my+nz+1)\left(\frac{dt}{ds}\right)^2$$

or

$$\frac{d^2x}{ds^2} = -\frac{l}{lx+my+nz+1}\left[1+\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2\right], \quad (32)$$

and analogous expressions in y and z.

As they do not contain derivatives of l, m, n, the acceleration is the same as if l, m, n were constant, *i. e.* as in the hyperbolic movement. We have found in this case the law of elliptic motion expressed by the equations (17) and (18) which may be applied with a change of orientation of the axis. These ellipses are in planes perpendicular to the plane $lx+my+nz+1=0$, and their small axis is in this plane. They are osculatory to the actual movement.

For rays of light, the ellipses are circles. The osculatory plane of the trajectory is at every instant t perpendicular to

the moving plane $lx+my+nz+1=0$, and the centre of curvature is in this plane.

5. *Movements of Class* B.

In the movements of class A it was possible to find in the moving system a time of simultaneity: *i. e.*, to give the same coordinate to all the points simultaneous with one another. We proceed to the case where this is impossible. Two arbitrarily given point-events may then be considered as simultaneous.

It is clear that if we determine the simultaneity on a twisting body, the earth for example, the simultaneous instants being the mean times of passage of light or Hertzian signals, two given instants may be considered as simultaneous if we make the determination along a path which makes a sufficient number of complete turns round the earth. As a matter of fact, two events which occur at the same place on the equator with a difference of proper time of 2×10^{-7} sec. are simultaneous along the equator. A. A. Michelson * has suggested a possible test of the rotation of the earth by interferential measures of simultaneity on the perimeter of a polygon, and experiments of this kind have been realized by G. Sagnac †, who succeeded in detecting the rotation of a twisting body by internal measures effected on it.

In these movements we use a parametric representation of the moving points, and the parameter t has no direct meaning of time.

Herglotz ‡ has shown that the movements of class B are one-parameter groups of movements in the universe, and he has determined from the theory of groups the four possible types of these movements.

I. Rotation of constant speed (elliptic group of Herglotz).

The equations of the movement are (λ constant):

$$\xi = x\cos\lambda t - y\sin\lambda t,$$
$$\eta = x\sin\lambda t + y\cos\lambda t,$$
$$\zeta = z,$$
$$\tau = t.$$

* Phil. Mag. viii. p. 716 (1904).
† *C. R.* clvii. pp. 708 & 1410 (1913).
‡ *Loc. cit.*

The corresponding interval may be written, using cylindrical coordinates:

$$ds^2 = -dr^2 - r^2 d\theta^2 - dz^2 - 2\lambda r^2 d\theta\, dt + (1-\lambda^2 r^2)\, dt^2. \quad (33)$$

The lines of simultaneity of direction conjugate to $dr = d\theta = dz = 0$ with respect to the light-cone are

$$-\lambda r^2 d\theta + (1-\lambda^2 r^2)\, dt = 0,$$

and the spatial distance obtained by elimination of dt is

$$d\sigma^2 = dr^2 + \frac{r^2 d\theta^2}{1-\lambda^2 r^2} + dz^2. \quad . \quad . \quad . \quad . \quad (34)$$

When r is a given function of θ the lines of simultaneity are

$$t - t_0 = \lambda \int_{\theta_0}^{\theta} \frac{r^2 d\theta}{1-\lambda^2 r^2}.$$

In the case of the earth, for $\theta - \theta_0 = 2\pi$, $\lambda = \frac{2\pi}{T}$,

$$t - t_0 \cong \frac{(2\pi r)^2}{T} = \frac{(2\pi r)^2}{T_{\text{sec.}} c^2} = \left(\frac{40}{300}\right)^2 \times \frac{1}{24 \times 60 \times 60}$$
$$= 2{\cdot}06 \times 10^{-7} \text{ sec.}$$

II. Hyperbolic group.

$$\xi = x + \lambda t,$$
$$\eta = y,$$
$$\zeta = z \cosh t,$$
$$\tau = z \sinh t.$$

The interval is

$$ds^2 = -dx^2 - dy^2 - dz^2 - 2\lambda\, dx\, dt + (z^2 - \lambda^2)\, dt^2. \quad . \quad (35)$$

For $\lambda = 0$, we obtain the hyperbolic movement which is the only one which belongs to both classes A and B.

The lines of simultaneity are

$$-\lambda dx + (z^2 - \lambda^2) dt = 0,$$

and the spatial distance

$$d\sigma^2 = \frac{z^2 dx^2}{z^2 - \lambda^2} + dy^2 + dz^2. \quad . \quad . \quad . \quad . \quad (36)$$

III. The third group (loxodromic group) is a combination of a rotation and an hyperbolic movement along the axis.

$$\xi = x\cos\lambda t - y\sin\lambda t,$$
$$\eta = x\sin\lambda t + y\cos\lambda t,$$
$$\zeta = z\cosh t,$$
$$\tau = z\sinh t.$$

The interval is

$$ds^2 = -dr^2 - r^2 d\theta^2 - dz^2 - 2\lambda r^2 d\theta\, dt + (z^2 - \lambda^2 r^2)\, dt^2. \quad (37)$$

The lines of simultaneity are

$$-\lambda r^2 d\theta + (z^2 - \lambda^2 r^2)\, dt = 0,$$

and the spatial distance

$$d\sigma^2 = dr^2 + \frac{r^2 z^2 d\theta^2}{z^2 - \lambda^2 r^2} + dz^2. \quad . \quad . \quad . \quad . \quad (38)$$

IV. The last group (parabolic group) depends on two constants, λ, μ.

$$\xi = x + \tfrac{1}{2}\lambda t^2,$$
$$\eta = y + \mu t,$$
$$\zeta = z + xt + \tfrac{1}{6}\lambda t^3,$$
$$\tau = \lambda t + r + xt + \tfrac{1}{6}\lambda t^3.$$

The interval is

$$ds^2 = -dx^2 - dy^2 - 2\mu\, dy\, dt + 2\lambda\, dz\, dt + (2\lambda x + \lambda^2 - \mu^2)\, dt^2, \quad (39)$$

and can be reduced, writing

$$x + \frac{1}{2\lambda}(\mu^2 - \lambda^2), \quad y, \quad \frac{1}{\sqrt{2\lambda}}(\mu y - \lambda z), \quad \frac{1}{\sqrt{2\lambda}} t$$

in place of

$$x, \quad y, \quad z, \quad t,$$

to the form

$$ds^2 = -dx^2 - dy^2 - 2dz\, dt + x\, dt^2. \quad . \quad . \quad . \quad . \quad (39')$$

The lines of simultaneity are

$$-dz + x\, dt = 0,$$

and the spatial distance

$$d\sigma^2 = dx^2 + dy^2 + \frac{dz^2}{x}. \quad . \quad . \quad . \quad . \quad (40)$$

It may be interesting to calculate the spatial curvature corresponding to these four types of movements.

For

$$d\sigma^2 = dx^2 + dy^2 + \phi^2 dz^2$$

the curvature is

$$\mathrm{R} = \frac{2}{\phi}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (41)$$

This formula may be applied to each of the four groups, and gives

$$\left.\begin{aligned} \mathrm{R}_1 &= 6\lambda^2(1-\lambda^2 r^2)^{-2}, \\ \mathrm{R}_2 &= 6\lambda^2(z^2-\lambda^2)^{-2}, \\ \mathrm{R}_3 &= 6\lambda^2(r^2+z^2)(z^2-\lambda^2 r^2)^{-2}, \\ \mathrm{R}_4 &= \tfrac{3}{2}x^{-2}. \end{aligned}\right\} \quad . \quad . \quad . \quad (42)$$

x and z are distances from a plane and r distance from a straight line. For R_3 this axis is perpendicular to the plane.

Writing $z = \lambda + x$ we find

$$\mathrm{R}_2 = \tfrac{3}{2}x^{-2}\left(1 + \frac{1}{2\lambda}x\right)^{-2},$$

which tends to R_4 when λ tends to infinity.

In the motions of class A the geometry was Euclidean, *i. e.* of vanishing curvature, and the space might be considered as a plane section, $t = \text{const.}$, of the four-dimensional universe.

But in the motions of class B there is no time of simultaneity and the space cannot be considered as a section of the Galilean universe. Moreover, the element of length along every line of simultaneity is expressed by the root of a differential quadric which defines a Riemann geometry of the above curvatures (42).

These curvatures of space are the only possible ones which can be produced by movement of the solid of reference in a Galilean field.

Cambridge,
March 6, 1924.

XX. *Adsorption from the Gas Phase at a Liquid-Gas Interface.*—Part II. *By* Thomas Iredale *.

In a previous communication (Phil. Mag. xlv. p. 1088 (1923)) the author described a method of measuring the surface tension of a liquid in contact with a vapour phase of varying composition, and studied in particular the change in the surface tension of mercury brought about by the adsorption of vapours of certain organic liquids. It was shown that, with a knowledge of the partial pressures of the vapours, and the use of the Gibbs adsorption equation, the excess of adsorbed vapour at the mercury surface could be calculated. It was thought remarkable that so few measurements of this interesting form of adsorption had ever been recorded in the literature of the subject; but it is noteworthy that as far back as 1877 Gibbs, in his original Memoir on Heterogeneous Equilibrium, suggested the possibility of calculating the amount of adsorption of water on mercury by noting the changes in the surface tension of the mercury under increasing partial pressures of water vapour.

Adsorption and Condensation.

In his original experiments the author generated vapours of different partial pressures by passing a slow current of dry air through organic liquids at different temperatures, and these air-vapour mixtures were then passed into a chamber at a somewhat higher temperature, where measurements of the surface tension of mercury were made. With methyl acetate vapour it was found that the surface tension fell rapidly with small partial pressures, but afterwards, as the pressure was increased, much more slowly, and became almost constant up to the point where the pressure was that of the saturated vapour. The curve indicating these changes was very similar to those usually obtained from data on the lowering of the surface tension of water by dissolving in it various "surface active" substances. It would seem in all these cases that the surface energy of the liquid ceases to change when the surface is saturated at a certain pressure or concentration with the molecules of the adsorbed substance. It was remarkable to observe, therefore, that when the vapour of the methyl acetate was saturated, great irregularities in the drop weight became apparent, the weight sometimes corresponding to what one would anticipate from the shape of the curve, namely, a surface tension of about

* Communicated by Prof. F. G. Donnan, F.R.S.

415 dynes per cm., and sometimes to as low a value as 365 dynes per cm., though intermediate values were also obtained. As the calculations of the surface tension were always based on the average weight of a number of drops, such intermediate values might reasonably have been thought to be the means of several extreme values, presupposing a fluctuation of conditions during any single determination. By experimenting with single drops however, these intermediate values were found to have a real significance, and some of the results obtained in this way are recorded in Table I.

Table I.

Weight of drop of mercury formed in saturated methyl acetate vapour at 26° C.	Surface tension.
0·1471 gram.	
0·1620 ,,	
0·1443 ,,	
0·1598 ,,	413 to 365
0·1628 ,,	dynes per cm.
0·1533 ,,	
0·1644 ,,	
0·1454 ,,	
0·1450 ,,	

One would be inclined to think that this was due to varying degrees of condensation of liquid methyl acetate on the mercury surface, but the precise conditions controlling this degree of condensation were still open to question. There seemed to be no exact relationship between the rate of drop formation—always greater than three minutes—and the rate of condensation. At the same time irregularities were almost certain to creep into the results, owing to the way in which the saturated vapour was employed. The reader is referred to the original paper for experimental details; but it may be remarked that when the temperature of the saturators was maintained at 26° C., that is, above room temperature, it became necessary to slightly warm the tube passing from the saturators to the thermostat containing the drop-weight apparatus, in order to prevent condensation of the vapour in this tube. This tended to make the saturators work irregularly, and the current of vapour was not always continuous. An improvement was effected by discarding the original saturators, and using a series of the ordinary bubbling type, and the temperature being raised slightly

above 26° C., the vapour began to condense on the walls of the glass dropping-chamber in the thermostat. The tendency of the vapour to condense on the mercury drop issuing from the pipette depended, of course, on the temperature fluctuation of its surface, and although the temperature fluctuation of the thermostat was never greater than 0°·2 C., and might be much less than this in the case of the mercury enclosed in the glass chamber, one could never be sure how irregular the temperature of the vapour might be after passing through the heated connecting tube. This effect, of course, would seem to be very slight, but it is possible that the rate of condensation of a saturated vapour is influenced very greatly by small changes of temperature and pressure, and the adjustment to uniform conditions in the thermostat might be slower than anticipated, even with very slowly moving currents. The author was therefore led by these considerations to discard such a method of producing a saturated vapour, and to study the condensation process under more static conditions. It seemed reasonable, also, to abandon the use of such a substance as methyl acetate, which has some objectionable characteristics. It was found extremely difficult to remove the last traces of its vapour from the dropping-chamber, as it is apparently hydrolysed after a time, with the formation of acetic acid, on the walls of the chamber. Dry air was passed continuously through, but the mercury surface was still contaminated with traces of the acetate or acetic acid, the surface tension never rising higher than 450 dynes per cm. Treatment with ammonia vapour was found to be useless, and the whole apparatus was then dismantled, thoroughly cleaned and dried, and remounted, after changing the lubrication of the taps, etc. Those parts of the apparatus requiring lubrication are, it must be admitted, one of its principal defects, unless a suitable lubricating agent be found. A preparation of dextrin dissolved in glycerin was originally used for this purpose, but certain samples of dextrin were found to be unsuitable, as they appeared to give off vapours quite appreciably influencing the surface tension of mercury. But the mercury surface was not found to be so sensitive to the presence of vacuum tap grease, nor to metaphosphoric acid, and the former of these materials was therefore used with water vapour, and the latter with organic vapours. In order to obtain the highest value for the surface tension in dry air, it was found necessary to do away with the lubrication altogether, and sometimes to pass the air through permanganate solution, soda lime, sulphuric acid, phosphoric

oxide, and cotton wool, before it entered the dropping-chamber ; and some samples of phosphoric oxide were found unsuitable for this purpose, owing, presumably, to the formation of some volatile compound. All this serves to show how sensitive the mercury surface is to the most minute traces of impurities adsorbed from the gaseous phase.

The Adsorption and Condensation of Water Vapour.

The presence of water vapour in the dropping-chamber was not found to influence the surface tension to any considerable extent, and this is due to the fact that its partial pressure at room temperature, even when saturated, is never very high. Consequently, when some determinations were carried out at room temperature (20° C.) outside the thermostat and in a space open to the atmosphere of the room, the degree of humidity of the atmosphere had very little effect on the results.

But when the vapour was saturated it was found possible, as in the case of methyl acetate, to actually condense the water on to the mercury surface, under certain conditions. The surface tension then fell to a very low value, corresponding to that found for the mercury-water interface.

Table II. records some of these measurements undertaken at 26° C., and it will be seen that, owing to the low pressures of water vapour used, the surface tension is still changing fairly rapidly with the pressure, but at higher pressures one would expect the σ–p curve to take the same form as the methyl acetate curve.

Table II.

Temperature of saturators.	Vapour pressure of water.	Drop weight.	Surface tension.
—	0 mm.	0·1903 gram.	472 dynes per cm.
13·7° C.	11·6 „	0·1857 „	461 „ „
20·0 „	17·5 „	0·1821 „	454 „ „
26·9 „	25·0 „ (saturated)	0·1793 „ to 0·1447 „	447 „ „ to 368 „ „

In the case of the unsaturated vapours, air was bubbled slowly through the saturators, and determinations of the surface tension were made not only when the air-vapour mixture was passing through the dropping chamber, but also when it was kept steady in the chamber, by closing the tap T_2 (see previous paper for diagram of apparatus), the correction to be applied for thermal transpiration being negligible.

The time of drop formation was never less than three and a half minutes, and when it was increased to ten minutes there was no appreciable change in the weight of the drop, so that it seems reasonable to suppose that the adsorption of the vapour is complete in this time.

In the case of the saturated vapour, however, a different procedure was adopted. The cup G was filled with water, and the ground glass joints of this part lubricated with the same liquid. The level of the water in the cup could be brought as near as desired to the drop issuing from the pipette. The temperature of the thermostat was then raised somewhat above 26° C. for a time, and on cooling to the original temperature with the tap T_1 closed, a slight blur on the walls of the dropping-chamber indicated condensation from a completely saturated vapour. As the walls of the chamber tended to cool faster than the pipette inside, condensation was more likely to occur on the walls than on the pipette, which was an advantage for the present purpose. As in the previous case of methyl acetate, variable results were obtained when the surface tension was measured in saturated water vapour (Table III.). But conditions were more under control than when the moving current of saturated vapour was used, and it was observed that the

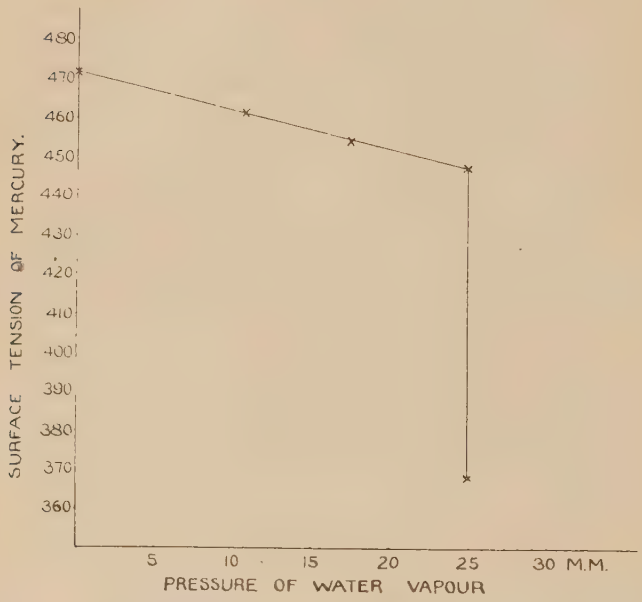
Table III.

Drop weights in grams, in saturated water vapour at 26° C.			Surface tension.
0·1515,	0·1642,	0·1574,	
0·1770,	0·1632,	0·1792,	
0·1448,	0·1603,	0·1704,	447 to 368
0·1782,	0·1770,	0·1665,	dynes per cm.
0·1477,	0·1767,	0·1742,	
0·1754,	0·1682,	0·1724.	

more slowly the drop was formed, and the nearer it was allowed to hang to the surface of the water in the cup, the smaller was its size, and the lower, therefore, the surface tension. The smallest values recorded in Table III. were obtained by allowing the drop to form about 1 mm. above the water surface, in a time varying from $3\frac{1}{2}$ to 20 minutes. When the period fell below $3\frac{1}{2}$ minutes, however (Table IV.), the drop weight increased very considerably, indicating a surface tension no less than that obtained by allowing the drop to form at a distance of over 1 cm. from the water

surface, which is what one would anticipate to be the surface tension in the saturated vapour from the curve (fig. 1) drawn with the aid of the data in Table II.

Fig. 1.



As a matter of fact, it was found almost impossible to condense the water on to the mercury surface if the drop were formed sufficiently distant from the water surface even if the period were as much as twenty minutes or more. These effects may be due, in part, to the variation in density of the vapour as we pass upwards from the water surface, but it is obvious that diffusion also plays a part in the phenomenon.

Table IV.

Period of drop.	Drop weight.	Surface tension.
$3\frac{1}{2}$ minutes to	0·1455 to	370 to
20 „	0·1448 gram.	368 dynes per cm.
2 „	$\left\{\begin{matrix}0{\cdot}1799\\0{\cdot}1793\end{matrix}\right\}$ gram.	447 „ „

The detachment of the drop had always to be very carefully watched, as certain irregularities were observed when any moisture condensed on the tip of the pipette, a

difficulty which could not always be obviated. The water sometimes tended to displace the mercury in the pipette, and when the drop broke off it was followed by a small secondary droplet, detached from the interior of the pipette just above the orifice. If the water formed a visible skin around the mercury, the drop was able to grow to a larger size than usual, owing to the additional support against gravity afforded by the skin. This is a rather important observation, as it shows that the invisible *condensed* film of water responsible for such a large decrease in the surface tension previously observed, must still be very thin compared with the dimensions of the drop itself. The surface tension measured in this case must really be the interfacial tension between the mercury and the water, and not the tension of the composite surface, mercury/water–water/air.

These results admit of a logical interpretation if we assume that there are two kinds of film in equilibrium with a liquid when another liquid tends to spread over its surface, The primary film, which may be only one or two molecules thick, is formed from the vapour of the spreading liquid. The secondary or condensed film, which is formed when the liquid has completely spread over the surface, may sometimes be much thicker than this, with corresponding differences in the interfacial tension. Some observations of Hardy (Proc. Roy. Soc. A, lxxxviii. p. 316 (1913)) lend support to these assumptions. He found that when a substance like benzene was spread over a water surface, lenses might be formed which were in equilibrium with a film of the benzene forming a composite surface with the water. This was possible because $T_B > T_A + T_{AB}$, where T_{AB} is the interfacial tension, and T_A, T_B, the surface tensions of benzene and water respectively. But with certain saturated substances where $T_B < T_A + T_{AB}$, spreading of this kind could not occur, though the vapours of these substances were able to condense on the surface, since $T_{AB} < T_B$. A film formed in this way could not have the properties of the liquid *en masse*, and no composite surface could be formed. The first of these conditions would hold in the case of water spreading on mercury (Harkins and Feldman, Jour. Amer. Chem. Soc. xliv. p. 2681 (1922)), since in this case $T_B > T_A + T_{AB}$, but in the author's mode of experiment the formation of the primary film preceded that of the condensed film, which, owing to diffusion or some other unavoidable factor, took a somewhat longer time for its completion.

In the present experiments which deal with the adsorption of molecules from the vapour phase, it is possible that the

primary phenomenon is the gradual formation of a unimolecular layer. The initial surface tension-vapour pressure curve would then correspond to the formation and perhaps, also, the compression of such a unimolecular layer (compare the work of Langmuir and Adam on the formation of unimolecular layers of fatty acids and other substances on a water surface). It must be remarked, however, that if the molecular attractive forces fall off very rapidly with distance, the effect on the surface tension might be due primarily to the first layer of molecules, and thus possibly the Gibbs equation might indicate a unimolecular layer, though in reality there were present in adsorption equilibrium a layer several or many molecules thick. After the vapour reaches the saturation value for a plane surface at a given temperature, the experimental data indicate that a new regime sets in.

In this case a very thin film of liquid may condense on the mercury surface, the thickness of which is not a determinate function of the pressure and temperature, though the most stable state corresponds to the formation of a film, which may, from the standpoint of intermolecular forces, be regarded as infinitely thick. In this region (saturated vapour) the experimental data indicate a very abrupt fall in the interfacial tension, the lowest value corresponding to the interfacial tension between mercury and liquid water in bulk. The formation of such a film will probably result in a still closer packing of the layer of molecules adjacent to the mercury surface.

The author is not aware that any experiments have been carried out on the compression of films on a mercury surface, but Hardy (Proc. Roy. Soc. A, c. p. 550 (1922)) has obtained some interesting results on lubrication, where primary films formed from organic vapours on a bismuth or glass surface were subjected to tangential forces. With the saturated vapour it was possible to obtain the same value for the coefficient of static friction as with a layer of the liquid itself. But these results are not quite in harmony with the author's for the following reason. There was a gradual change in the friction as the vapour pressure approached the saturation value, whereas the surface tension was always found by the author to change abruptly in the saturated vapour in the several cases examined (water, methyl acetate, benzene, etc.), even when an adsorption maximum was indicated by the curve at pressures much below the saturation pressure. The molecules on the mercury surface were not, of course, subjected to any external forces while

the drop was forming, but the two phenomena differ also in some other respects. In the experiments on static friction the vapour pressure of the lubricant had to exceed a definite small value before any change in the friction could be observed. The surface tension of the mercury, on the other hand, is sensitive to the most minute traces of organic vapours, at vapour pressures presumably very much lower than this limiting value in the friction experiments. There possibly is a limiting pressure for the surface tension, but it is too small to be appreciable in the present experiments.

Another important factor is the time of formation of the primary film. Hardy found with some substances that the molecules took twenty minutes or more to orient themselves on the surface. The author has always observed, on the other hand, that the adsorption of *unsaturated* vapours on the mercury surface is a very rapid process, and the maximum lowering of the surface tension occurs in less than a minute. The somewhat longer time necessary to form the *condensed* film from the saturated vapour must depend more on the rate of diffusion of the molecules to the surface, and hence the presence of the air and the distance of the mercury drop from the water surface must be the principal factors in the process. To test these points further the author has begun some experiments on the adsorption of vapours in the absence of air, which will be discussed later.

A few words may be said here about the work of Cantor (Wied. *Ann.* lvi. p. 492 (1895)) which has a direct bearing on the present experiments. Cantor argued that it was possible to condense the vapour of a liquid on to the surface of another liquid at a pressure less than the saturation pressure, owing to the decrease in free energy at the surface. The particular equation he derived is true only in so far as the pressure is independent of the thickness of the condensed film; but he afterwards ignores this assumption in deriving the equation for the tension of the composite surface. The argument is faulty and is based on incomplete experimental evidence. From the author's results it would seem that the transition from an adsorbed film, possibly only one molecule in thickness, to a condensed film having the properties of the liquid in bulk, is more abrupt than was to be anticipated on any such theory as Cantor's.

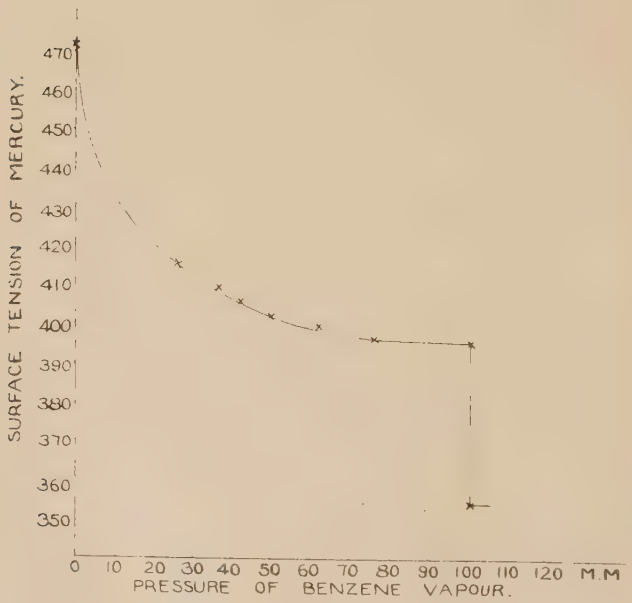
Cantor considered that, because the vapours of benzene and amyl alcohol generated at 21° C. were able to lower the surface tension of mercury at 23° C., there were good grounds for supposing that condensation of the vapours had occurred at a temperature above the saturation temperature,

and consequently below the saturation pressure. These results do not necessarily indicate that condensation has occurred, and so far as the author can see from the actual values of the surface tension change, they merely show the existence of a primary "gaseous" film.

The Adsorption and Condensation of Benzene Vapour.

Of the other substances studied the most interesting was benzene. A pure specimen of this liquid was obtained by freezing a good thiophene-free sample that had been used for cryoscopic purposes. The crystals were melted and the liquid distilled, the portion boiling at 80° C. being collected. It was afterwards dried over phosphoric oxide and again distilled.

Fig. 2.



The results of experiments with benzene vapour are recorded in Table V., and the curve in fig. 2 was constructed with the aid of these data. It is similar in form to the previous curve obtained with methyl acetate, and illustrates an apparent adsorption maximum before condensation of the vapour takes place.

The results in Table VI. show that, as in the case of water vapour, the condensation of benzene under static conditions takes some minutes for its completion.

The Reversibility of the Phenomenon.

When the drop issuing from the pipette had grown to a certain size in the vapours of water or benzene, and was about to break off, this detachment could sometimes be retarded by dexterously forcing a current of dry air into the dropping-chamber, and sweeping out the vapour. The fact that the drop could afterwards grow to a larger size before detaching is some evidence for the belief that the phenomenon is a reversible one, as the surface tension was evidently increased by desorption of the vapour.

TABLE V.

Temperature of saturators.	Vapour pressure of benzene.	Drop weight.	Surface tension.
—	0 mm.	0·1903 gram.	472 dynes per cm.
0° C.	26·5 „	0·1668 „	415 „ „
5·4 „	37·0 „	0·1636 „	410 „ „
8·1 „	42·1 „	0·1610 „	406 „ „
11·2 „	49·9 „	0·1592 „	402 „ „
15·5 „	61·8 „	0·1583 „	400 „ „
20·0 „	76·3 „	0·1565 „	396 „ „
26·0 „	101·0 „ (saturated vapour).	0·1562 „ to 0·1416 „	395 „ „ to 354 „ „

TABLE VI.

Condensation of benzene on mercury near the benzene surface.

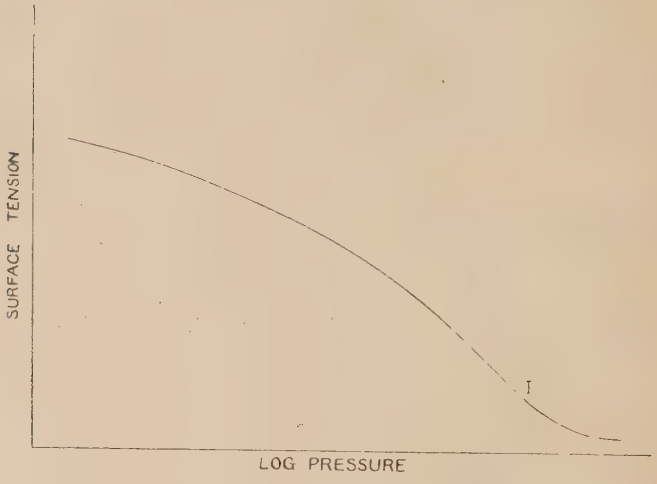
Period of drop.	Drop weight.	Surface tension.
2 minutes	0·1522 grams.	388 dynes per cm.
5—20 „	0·1416 „	354 „ „

The Calculation of Γ *from the* σ–p *Curve.*

The tangent to the σ–p curve, the value of $\frac{d\sigma}{dp}$, enters into the Gibbs equation for calculating the value of Γ, the excess surface concentration of vapour. Such a calculation is meaningless, however, when the curve becomes asymptotic to the line of lowest surface tension, as the tangent gradually approaches a zero value. All we are concerned with is the maximum value of Γ, and that such a maximum exists is evident from the following considerations.

The Gibbs equation, $\Gamma = -\rho\frac{d\sigma}{dp}$, since ρ is proportional to p, may be written in the form $\Gamma = -k\frac{d\sigma}{d\log p}$, where k is a constant. Taking the values of $\log p$ for the abscissæ, and those of σ for the ordinates, the usual form of curve would be as shown in fig. 3. The value of $\frac{d\sigma}{d\log p}$ passes

Fig. 3.



through a maximum at the point of inflexion I, and so also must the value of Γ be a maximum at this point. The theory, of course, is quite unable to predict the shape of the curve, but in making any calculation of Γ, especially when we desire to know the actual value of the saturated adsorption, it is important to bear this point in mind, though most investigators seem hitherto to have ignored it.

In the case of water vapour at the pressures examined, the curve (fig. 1) does not indicate a maximum value of the surface adsorption, which is still changing fairly rapidly with the pressure, just before the saturation point.

The value of Γ at the saturation point was found to be $1{\cdot}8\times10^{-8}$ gram per sq. cm., which is somewhat less than that required for a monomolecular film (3×10^{-8}; see also Langmuir, Jour. Amer. Chem. Soc. xl. p. 1379 (1918)).

The maximum value of Γ for benzene was found to be $6{\cdot}2\times10^{-8}$ gram per sq. cm. From this it can be calculated that the area per molecule on the surface is 21×10^{-16} sq. cm. This is very near the value $23{\cdot}8\times10^{-16}$ sq. cm. which

Adam (Proc. Roy. Soc. A, ciii. p. 676 (1923)) gives for the area per molecule of certain benzene derivatives on the surface of water.

There seems to be some evidence, therefore, for the assumption that vapours adsorbed on the surface of mercury tend to form primary monomolecular films.

The Surface Tension of Mercury in a Vacuum.

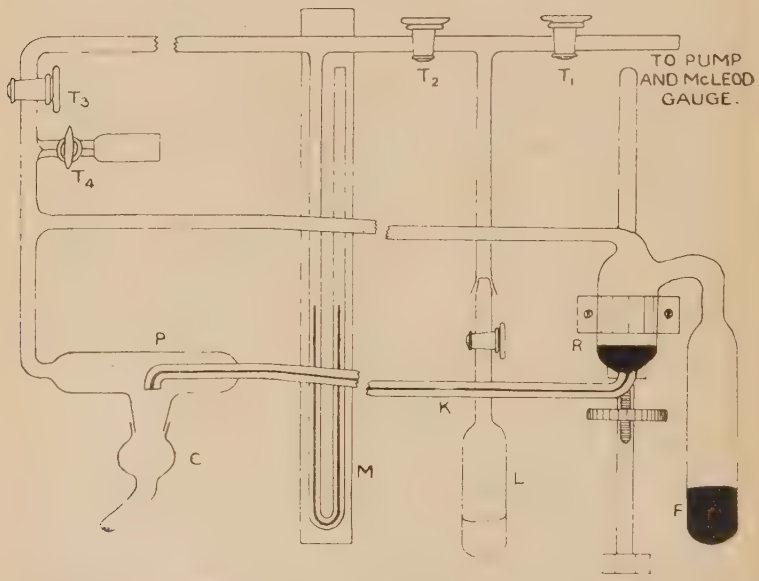
There exist some discrepancies in the literature dealing with the measurement of the surface tension of mercury in a vacuum. The results of Stockle (Wied. *Ann.* lxvi. p. 499 (1898)), which are more often quoted than any others, show some remarkable anomalies. He found that the surface tension in air and certain gases was always higher than in a vacuum, but it fell after a time to the same value. More recent experiments, those of Hogness (Jour. Amer. Chem. Soc. xliii. p. 1621 (1921)), Harkins and Ewing (*ibid.* xlii. p. 2539 (1920)), Palacios and Lasala (*Anal. Fis. Quim.* xx. p. 505 (1922)), do not confirm these results, and these workers found that the surface tension was either the same in a vacuum as in dry air and other indifferent gases, or, if anything, a little higher. One would expect the adsorption of gases to lower the surface tension, and would be inclined to think that these later experiments were more reliable, though Stockle's experiments seemed to be carried out with no less care and precaution. It was obvious to the author that this point must be finally settled before continuing investigations on the adsorption of vapours.

Devices for measuring the surface tension of mercury in a vacuum by the drop-weight method have been described by Cenac (*Ann. de Chim. Phys.* xxix. ser. 8, p. 298 (1913)) and Harkins and Ewing (*loc. cit.*), but they were not found wholly suitable for the present purpose. In addition to this type of measurement, the surface tension in vapours unmixed with air had to be measured at different pressures and temperatures, and the device had also to allow of a varying period of drop formation—a few seconds to an hour or more,—which considerably increased the difficulties of the technique.

The original apparatus went through several stages in its development, but the final form adopted will be readily understood from the diagram (fig. 4). The distance between the dropping-chamber P and the container R was about four feet, giving considerable flexibility to the device, and permitting the container to be raised and lowered through a

distance of several centimetres. The level of the mercury in the container was always higher than the column of mercury throughout the length of the capillary tube, so that no part of the column was under a negative pressure when the apparatus was evacuated. The minute capillary K was introduced some distance down the tube, so as to be on the same level as the orifice. These precautions were taken to minimize the danger of the column detaching at any point, as great difficulty was experienced in getting regular drop formation in some of the earlier apparatus, owing to defects

Fig. 4.



of this kind. The tip of the pipette was made in the manner described in the previous paper, and the diameter measured with the aid of a travelling microscope. It was necessary for the orifice to be only one centimetre from the bend in the tube, and some of the tubes had to be discarded on this account, as the bending tended to make the capillary elliptical in cross-section.

The apparatus was evacuated by means of a two-stage mercury vapour pump, operated by a Toepler pump. This arrangement helped to promote a quiet and regular ebullition of the mercury in the vapour pump, and hence to minimize vibration effects.

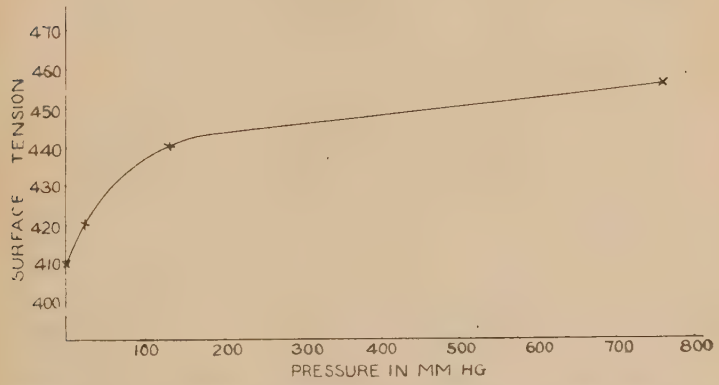
In the first experiments pure mercury was transferred to F, and distilled over into R at a pressure of 10^{-5} mm. In removing the cup C, air from the room passed into the apparatus, in addition to air already admitted by the tap T_4, and then some rather remarkable results were obtained.

TABLE VII.

Temperature 18–19° C.	Surface tension.
In air, pressure 763 mm.	456 dynes per cm.
,, 130 ,,	440 ,, ,,
,, 24 ,,	420 ,, ,,
,, 10^{-4} ,,	410 ,, ,,

As the pressure of the air in the apparatus was decreased, the surface tension gradually fell, and diminished very rapidly as the pressure became very low (Table VII. and fig. 5). It seemed as though it were a selective adsorption

Fig. 5.



effect, some impurity tending to adsorb on the mercury surface more readily in a vacuum than in air. Such an impurity would obviously be of a condensable kind, not readily detectable by the McLeod gauge, and might be water vapour—always the last gas to be removed from any vessel,—or some volatile compound from the lubricant of the taps, or from the phosphoric oxide used for drying purposes. The effects were apparent, however, whether the phosphoric-oxide tube were inserted or not, and a change of lubricant from rubber grease to meta-phosphoric acid did not alter the results. Attempts were made to do away with the lubricant altogether and to use mercury seals, the manometer M and the flask L were removed, and a liquid-air trap inserted

between the mercury-vapour pump and the dropping-chamber. These changes, however, were without effect, and it would seem that the impurity was coming from the glass or from the mercury itself. The apparatus was then dismantled and thoroughly re-cleaned, and dried in a current of hot air. The mercury was re-distilled, twice in a current of air, and then *in vacuo*. Determinations were then made as quickly as possible after distilling the mercury into the container. The surface tension was found under these conditions to be 475 dynes per cm. at a pressure of 10^{-5} mm. On admitting air it fell to about 460, and on re-evacuating to 10^{-5} mm. it decreased, as before, to 410 dynes per cm.

TABLE VIII.

		Period of drop.	Surface tension.
In air, atmospheric pressure			
	760–750 mm.	30 seconds.	456 dynes per cm.
		5–10 minutes.	450–448 ,, ,,
		1–2 hours.	450–446 ,, ,,
pressure	13 mm.	10 seconds.	460 ,, ,,
		1 minute.	440 ,, ,,
		5–10 minutes.	427 ,, ,,
pressure	10^{-4} mm.	20 seconds.	417 ,, ,,
		5–10 minutes.	412 ,, ,,

It was interesting to observe also that the impurity was adsorbed more quickly in a vacuum than in air (Table VIII.).

Hogness (*loc. cit.*) states that a thorough preliminary heating of his apparatus was necessary in order to obtain the maximum value for the surface tension. This was presumably to drive out all the condensed water from the system, which the vacuum pump, of itself, was unable to do. Harkins and Ewing (*loc. cit.*) merely state that the pressure in their vacuum was so low as to be undetectable by the McLeod gauge (below 10^{-7} mm.). These workers did not observe the interesting phenomenon described in this paper, but they seem to have adjusted conditions whereby it would be overlooked.

On admitting water vapour into the dropping-chamber—temperature 18° C. and pressure 15·5 mm.—the surface tension was found to be 411 dynes per cm. The vapour was saturated, but did not readily condense on the mercury surface. This is much the same as the surface tension *in vacuo*, and it is possible that the adsorption of water vapour

from the glass of the apparatus is responsible for the anomalies that have been observed.

It is possible, also, that the presence of air may raise the pressure of water vapour necessary to form a saturated, adsorbed film, but this by no means invalidates any of the conclusions arrived at in the earlier part of this paper. At the same time, it seems remarkable that an adsorbed film of water at ordinary temperatures can exist on a mercury surface at so low a pressure as 10^{-5} mm. This matter is being further investigated, but it would seem that similar effects to those described were operative in Stockle's experiments, and were responsible for the anomalies he observed.

I am indebted to Professor Donnan for the great interest he has taken in this investigation, and to the Royal Commissioners of the 1851 Exhibition for a scholarship which is enabling me to continue the work.

The William Ramsay Laboratories of
Inorganic and Physical Chemistry,
University College, London,
April 23, 1924.

Errata.

Phil. Mag. xlv. 1923:—

Page 1090, lines 13, 14, read: "any liquid of known density."

Page 1093, line 8, read: "density of benzene at this temperature is 0·879."

XXI. *Adsorption Forces and their Electrical Nature.* *By* Professor Boris Iliin, *the Physical Institution of the Science Institution of Moscow* *.

[A Preliminary Report.]

THE existence of a close relation between adsorption and the phenomena of surface tension, the phenomena of the potentials of electrodes, the phenomena of electrocapillarity, the coagulation of colloidal solutions under the influence of electrolytes, leads to the problem of the electrical nature of the forces of adsorption. But if one considers the potential of adsorption as electric, one is able to look for definite relations between electrical constants and the adsorption capacity $C_\sim$ and the heat of adsorption Q, because the relation of $C_\sim$ and Q to the adsorption potential

* Communicated by the Author.

can be established by aid of the reasonings of Eucken * which led him to the formula:

$$C_\sim = \frac{S\sigma n_0 e^{\frac{c}{T}}}{\mu \frac{c}{T}}.$$

Here S is the surface of adsorption.

$\sigma = \rho_1 + \rho_2$, where ρ_1 and ρ_2 are the radii of the molecules of the adsorbent and the adsorbed substance respectively.

$C = \frac{a}{\sigma^\mu} R = -\frac{K_\sigma}{R}$, whereas K_σ is a private quantity for the potential of attraction $K = \frac{a}{r^\mu}$ for $r = \sigma$.

Using the same method †, and considering the potential of attraction as an electric potential, I arrived at the relation:

$$C_\sim = Sn_0 \frac{RTr_0^\nu}{b}\left[e^{\frac{b}{RTr_0^\nu}(r_0-\sigma)} - 1\right], \quad . \; . \; . \quad (1)$$

where r_0 is the width of the adsorption film, and "b" enters into the expression for the force of adsorptive attraction $F = \frac{b}{r^\nu}$ and is defined by the position of electrical charges of electrical fields of the adsorbent and of the adsorbed substance ‡.

For the case of the adsorption of ions

$$b = b'\kappa\eta,$$

where η is the elemental electric charge, κ the valency of the ions, b' a constant.

Therefore

$$C_\sim = S\eta_0 \frac{RTr_0^\nu}{b'\kappa\eta}\left[e^{\frac{b'\kappa\eta}{RTr_0^\nu}(r_0-\sigma)} - 1\right], \; . \; . \; . \quad (2)$$

a relation which gives an explanation of the strong augmentation of the coagulating action with addition of electrolytes of higher valency ‡.

* Eucken, *Ber. d. deutsch. Phys. Ges.* xvi. p. 345 (1914).

† For a more general case, when adsorption is not limited by the surface densification alone, I developed a general theory of adsorption processes, tested by experiments. *Vide* Iliin, 'Reports of the Physical Institution of the Science Institution of Moscow,' i. (1921); Iliin, *Zeit. f. phys. Chem.* cvii. p. 145 (1923).

‡ Iliin, 'The Electrical Nature of the Adsorption Forces': a Report to the Physical Institution of the Moscow Sci. Inst. Dec. 1920; "On the Theory of Absorption Processes," Proceedings of the 3rd Mendelejev Congress at Petrograd, May 1922.

The aim of this work is, besides the equations (1) and (2), to find even definite relations between the adsorptive constants C and Q and the dielectric constant ϵ of the adsorbed gas—considering not the *adsorption of ions* but the adsorption of neutral molecules, and considering the neutral molecule as a compound of positive and negative charges according to contemporary conceptions of electron-theory displaced under the influence of the external electrical field of the adsorbent.

In fact, if the forces of adsorption are of electro-statical origin, then the tension of the field of the adsorbent E placed in an adsorbed gas is defined by a known relation:

$$E = E_0 - 4\pi N d\eta',$$

where $d\eta'$ is defined by a displacement of charges in the gas molecule under the influence of the field; d is the distance between the displaced charges, η' the size of the charge, N the number of the charges of both signs in a unit of volume; E_0 the tension of the field of adsorption in a vacuum; $E = \frac{E_0}{\epsilon}$ the tension of the field of adsorption in a gas with a dielectric constant ϵ.

Therefore

$$\tau = d \cdot \eta' = \frac{E_0}{4\pi N}\left(\frac{\epsilon - 1}{\epsilon}\right). \quad . \quad . \quad . \quad . \quad (3)$$

For the tension of a field of adsorption, leaving the question of the distribution of the electric charges on the adsorbent open, we make use of the expression

$$E_0 = \frac{e_1}{r^\nu},$$

where e_1 characterizes the electric density on the adsorbent. For the force of the field of adsorption acting on the neutral molecule, we have

$$F = e_1 \cdot \eta'\left(\frac{1}{r^\nu} - \frac{1}{(r+d)^\nu}\right) = e_1 \cdot \eta' \frac{\nu \cdot d}{r^{\nu+1}},$$

and considering (3),

$$F = e_1 \cdot \nu \frac{\tau}{r^{\nu+1}} = \frac{e_1 \nu}{r^{\nu+1}} \frac{E_0}{4\pi N}\left(\frac{\epsilon - 1}{\epsilon}\right) \quad . \quad . \quad . \quad (3')$$

for the given case "b" in the formula (1) is equal to

$$e_1 \cdot \nu \frac{E_0}{4\pi N}\left(\frac{\epsilon - 1}{\epsilon}\right) \cdot r^{-1},$$

and substituting this expression in the formula (1), we obtain

$$C_\sim = S \cdot \eta_0 \frac{RT r_0^{\nu+1}}{e_1 \nu \frac{E_0}{4\pi N}\left(\frac{\epsilon-1}{\epsilon}\right)} \left[e^{\frac{e_1 \nu \frac{E_0}{4\pi N}\left(\frac{\epsilon-1}{\epsilon}\right)}{RT r_0^{\nu+1}}(r_0-\sigma)} - 1 \right], \quad (4)$$

or in rejecting the unity in comparison with

$$e^{\frac{e_1 \nu \frac{E_0}{4\pi N}\left(\frac{\epsilon-1}{\epsilon}\right)}{RT r_0^{\nu+1}}(r_0-\sigma)},$$

we have

$$\log\left\{C_\sim \cdot \left(\frac{\epsilon-1}{\epsilon}\right)\right\} = \log A + B\left(\frac{\epsilon-1}{\epsilon}\right)\log e,$$

where A and B are expressions no longer containing the dielectric constant ϵ. The following Table I. (Graph 1) gives corresponding experimental data :—

TABLE I. (Graph 1.)

Dielectric Constant ϵ and the Adsorption Capacity $C_\sim$.

Adsorbed gas.	Dielect. constant ϵ.	$\log C_\sim$. Data of I. H. Homfray.	$\log C_\sim$. Data of A. Titoff.	$\log\left\{C_\sim\left(\frac{\epsilon-1}{\epsilon}\right)\right\}$. Data of I. H. Homfray.	$\log\left\{C_\sim\left(\frac{\epsilon-1}{\epsilon}\right)\right\}$. Data A. Tit
H_2	1·000264	.....	−0·644	.....	−4·2
Ar	557	−1·11+3	.....	−1·36	.....
N_2	581	−0·92+3	+0·37	−1·16	−2·8
CO	695	−0·70+3	.....	−0·86	.....
CH_4	944	−0·25+3	.....	−0·27	.....
CO_2.....	985	+0·20+3	+1·48	+0·19	−1·5
C_2H_4	1312	+0·58+3	.....	+0·70	.....

The linear interrelation between $\log\left\{C_\sim\left(\frac{\epsilon-1}{\epsilon}\right)\right\}$ and $\frac{\epsilon-1}{\epsilon}$ required according to the theory is confirmed.

Because the heat of adsorption Q is defined by the potential of attraction, it is evident that according to the

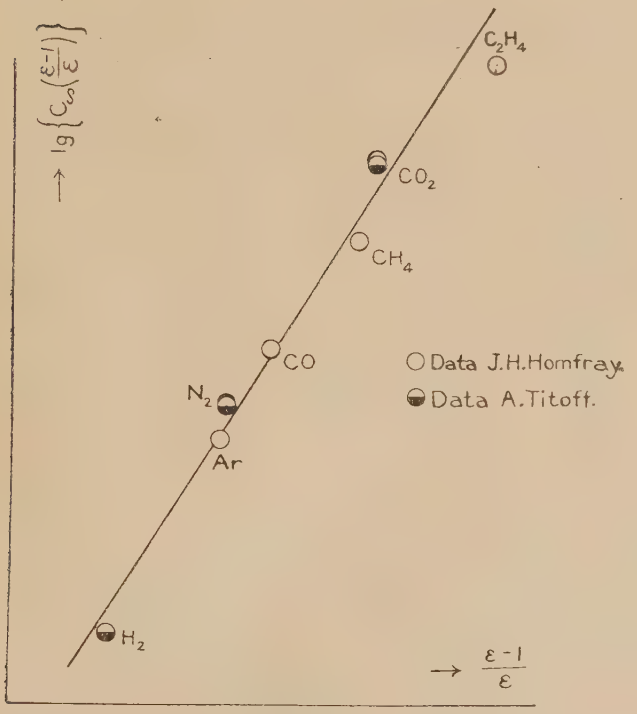
above-mentioned considerations a definite relation must exist between Q and ϵ*.

In a vacuum the adsorbent (also cool) has an external energy of a field of adsorption whose density is expressed by

$$W_0 = \frac{E_0^2}{8\pi},$$

where E_0 is the tension of the field of adsorption in a vacuum.

Graph 1.



If the adsorbent is placed in a gas with a dielectric constant ϵ, then the density of the energy of the field of adsorption has to change into

$$W_1 = \frac{\epsilon E^2}{8\pi},$$

where $E = \frac{E_0}{\epsilon}$ is the tension of the field of adsorption in a given gas with a dielectric constant ϵ.

* The deduction and the calculations concerning Q were made by my co-worker, Mr. B. B. Tarasov, who is working with a great amount of material and whose work is shortly going to press.

The difference $W_0 - W_1$ defines the specific heat of absorption Q :

$$Q = W_0 - W_1 = \frac{E_0^2}{8\pi} - \frac{\epsilon E^2}{8\pi} = \frac{E_0^2}{8\pi}\left(\frac{\epsilon - 1}{\epsilon}\right).$$

The same result can be obtained if one considers Q as a potential of the force of adsorbing attraction :

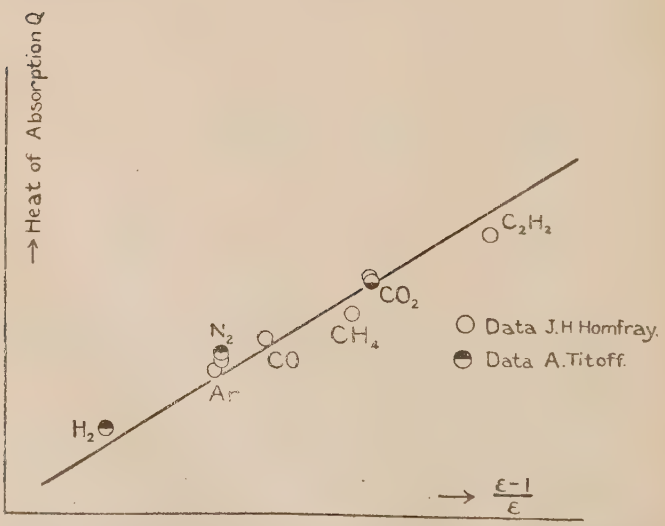
$$F = \frac{e_1 \cdot \nu}{r^{\nu+1}} \frac{E_0}{4\pi N}\left(\frac{\epsilon - 1}{\epsilon}\right).$$

TABLE II. (Graph 2.)

Dielectric Constant ϵ and the Heat of Adsorption Q.

Adsorbed gas.	Dielectric constant ϵ.	The Heat of Adsorption, Q. Data of I. H. Homfray.	Data of A. Titoff.
H_2	1·000264	.....	2408
Ar	557	3993	.....
N_2	581	4270	4460
CO	695	4860	.....
CH_4	944	5570	.....
CO_2	985	6560	6467
C_2H_4	1312	7800	.....

Graph 2.



Experimental data prove the proportionality between Q and $\frac{\epsilon - 1}{\epsilon}$ as required by the theory quite satisfactorily,

especially if one takes into consideration considerable mistakes in the definition of the heat of adsorption and of the dielectric constants.

As Debye * has shown, the constant of attraction in the equation of van der Waals is nothing else but the electrostatic energy of the molecular attraction, or, according to our conceptions, a quantity related to $\frac{\epsilon-1}{\epsilon}$. Without establishing any definite law, we give here items which show a quite definite parallelism in the growth of $\log C_\sim$, Q, $\frac{\epsilon-1}{\epsilon}$, and of the constant of van der Waals.

TABLE III.

Attraction-Constant of van der Waals and Adsorption Constants.

rbed	Dielect. const. ϵ.	$\log C_\sim$. Data of I. H. Homfray.	$\log C_\sim$. Data of A. Titoff.	Q. Data of I. H. Homfray.	Q. Data of A. Titoff.	Attraction-constant of van der Waals.
....	1·000264	.....	−0·644	.....	2408	0·00042
....	557	−1·11+3	.....	3993	.....	259
....	581	−0·92+3	+0·37	4270	4460	268
....	695	−0·70+3	.....	4860	.....	275
...	944	−0·25+3	.....	5570	.....	367
...	985	+0·20+3	+1·48	6560	6467	699
...	1312	+0·58+3	.....	7800	.....	833

Conclusions.

The general results of this work are as follows :—

(1) On the basis of a conception of an electric nature of the forces of adsorption, we have obtained a relation between the adsorption capacity $C_\sim$ and the dielectric constant of the adsorbed gas ϵ in the form

$$C_\sim = S \cdot \eta_0 \frac{RTr_0^{\nu+1}}{e_1 \frac{E_0}{4\pi N}\left(\frac{\epsilon-1}{\epsilon}\right)} \left[e^{\frac{e_1 \nu \frac{E_0}{4\pi N}\left(\frac{\epsilon-1}{\epsilon}\right)}{RTr_0^{\nu+1}}(r_0-\sigma)} - 1 \right],$$

which is proved by experimental data.

(2) On the basis of the same conceptions, the heat of adsorption Q is defined by the difference between the external energy of the adsorption field of the adsorbent in a vacuum

* Debye, *Phys. Zeit.* xxi. p. 178 (1920).

and in a given gas with a dielectric constant ϵ, and for Q we have obtained the equation :

$$Q = \frac{E_0^2}{8\pi}\left(\frac{\epsilon-1}{\epsilon}\right).$$

This relation is in good accordance with experimental data.

(3) A definite parallelism in the changes of $\log C_\infty$, Q, $\frac{\epsilon-1}{\epsilon}$, and of the constant of van der Waals is established.

Finally I have to express my deep gratitude to the Member of the Russian Academy of Science, Professor P. P. Lasareff, the Director of the above-named Institution, where some of my work and that of my co-workers is conducted, for the lively interest and attention given by him to it.

Physical Institution of the
Science Institution of Moscow,
11th August, 1923.

XXII. *The Number of Alpha-Particles emitted by Radium.* *By* Victor F. Hess, *Ph.D.*, *and* Robert W. Lawson, *D.Sc.*, *F.Inst.P.**

IN a recent communication (*Zeit. f. Physik*, vol. xxi. pp. 187–203, 1924), H. Geiger and A. Werner published the results of a redetermination by the scintillation method of the number of alpha-particles emitted per second from 1 gram of radium, and find this number to be $3{\cdot}4 \cdot 10^{10}$. Although this value agrees with that originally found by E. Rutherford and H. Geiger (Proc. Roy. Soc. (A), lxxxi. p. 141, 1908), it differs from the corrected value $3{\cdot}57 \cdot 10^{10}$ given by E. Rutherford (Phil. Mag. (6) xxviii. p. 320, 1914), and still more from that found later by the present writers, viz. $3{\cdot}72 \cdot 10^{10}$ (*Wien. Ber.* cxxvii. p. 405, 1918). In view of this discrepancy, perhaps we may be permitted to make some comment on Geiger and Werner's results, more especially as in our earlier work on this subject we also obtained results comparable with those of Geiger and Werner, and only after protracted experimenting and numerous improvements in our apparatus were we able to overcome

* Communicated by the Authors.

the difficulties involved and arrive consistently at the higher figure. After careful consideration of the available evidence, we still feel that the value $3{\cdot}72 \cdot 10^{10}$ cannot be far removed from the true value, and we propose in what follows to state our main reasons for concluding that Geiger and Werner's value is too low.

Radioactive Preparations.

In consequence of the difficulties connected with the short life of RaC, Geiger and Werner decided in favour of small conical emanation containers with thin mica window, similar to those previously used in the Manchester laboratory, but of metal instead of glass. The conical part was of highly polished brass (vol. ca. 10 mm.^{3}), silver-soldered at the apex to a fine platinum tube (bore 0·1 mm., length ca. 5–6 mm.), to which was fused at the remote end a glass capillary tube, which led to the emanation supply. After filling the small conical container with emanation and mixed gases at a pressure of a few cm., it was sealed off by fusing the glass round the remote end of the platinum tube, so that all the emanation was contained in the brass cone and the platinum tube (fractional volume ca. 1 per cent.). A small correction could be added to take account of this inoperative volume of the platinum tube.

To the present writers it appears that the use of such containers is open to question. As is well known, platinum usually contains quite an appreciable amount of occluded gases, which can be partially removed by heat, even in boiling water. On cooling the platinum in a gas-filled space, it will re-occlude gas, rapidly at first, and then more and more slowly with the lapse of time. Now the platinum in Geiger and Werner's emanation containers would necessarily become appreciably heated in the sealing-off operation, and perceptible quantities of occluded gas would enter the container, which was at reduced pressure. On cooling, the platinum will re-occlude gas from the container, and some of the emanation will disappear in this way. It is only necessary to assume that ca. 1/20 mm.^{3} (at N.P.T.) of the mixed gases disappear in this manner to account for the 10 per cent. discrepancy between the results of Geiger and Werner and our own, for whereas the occluded emanation will be partially effective in the emanation measurement, it will be wholly ineffective in the counting experiments. Although this effect may not be as large as 10 per cent., it is probably appreciable, and ought to be detectable by means of a gamma-ray electrometer.

Where continuous measurements are available, we have plotted as ordinate the number (Z) of alpha-particles per 1 gm. Ra per sec., and as abscissa the strength of the preparation in millicuries. The average graph through the points for the RaC′ determinations with preparations K and O (Geiger and Werner) are in conformity with the view that the emanation (and mixed gases) was removed (in addition to disintegration) at the rate of ca. 1 per cent. per day, and the RaEm determinations with preparations O and L, though less definite, also indicate a diminution with time. The remainder of their measurements are insufficient in number to apply this test. Whereas this might be accounted for by assuming that their mica window was not perfectly air-tight, it seems more probable that it can be explained by a slow occlusion by the platinum of the mixed gases in the emanation container. In view of this result, it seems feasible to suggest that several per cent. of the emanation was involved in the initial occlusion by the platinum on cooling (see previous paragraph).

It is rather unfortunate that Geiger and Werner in general only measured the Ra-equivalent of their preparations once, and then calculated the amount of emanation present later by using the known decay curve of the gas. Repeated measurement would have been valuable from the point of view of the above considerations.

It is interesting to note that their result with Em + A + C′ ($3{\cdot}39_1 . 10^{10}$) is about 1·6 per cent. greater than that ($3{\cdot}33_6 . 10^{10}$) obtained by using RaC′ alone. Whereas the emanation is contained in the volume of the container (which is at low pressure), the active deposit (RaA, RaB, RaC) will be buried in the walls of the vessel by virtue of recoil phenomena, and at depths varying from nil to the double range of the recoil atoms (RaA and RaB) in the material of the walls. Slight scattering and absorption effects would appear to be inevitable, and the correction for this is probably nearer 2 per cent. than 1 per cent., the amount suggested by Geiger and Werner.

Scintillation Screens.

Great care was taken by Geiger and Werner in the preparation of the material for their screens. They tested samples crystal by crystal and found that about 98 per cent. (590 in 600) gave scintillations with alpha-rays. Taking account of the size of the inactive crystals they estimate that their material would be about 99·6 per cent. efficient. These

tests, however, seem to have been made on loose material, and it is very doubtful whether they would have obtained similar efficacy had they tested the crystals after moistening, even with weak resin or turpentine oil, which substances were used as adhesives in coating the screens with zinc sulphide. It would appear very probable that the use of these binding agents will result in a diminution of the scintillating power of the material. It is a well-known fact that binding agents, even water-free varnishes, reduce the surface luminosity of zinc sulphide mixed with radioactive material emitting alpha-rays (luminous substance) to 30 per cent., and even less, of their original luminosity. This is certainly not due to a reduction of the intensity of the individual flashes so much as to a reduction of the number of effective hits.

Thus, in spite of their having estimated the percentage area of the surface of the screens uncovered by zinc sulphide, it is very doubtful whether Geiger and Werner are justified in assuming their screens to have 100 per cent. efficiency. Hitherto, a screen with an efficiency of 90 per cent. has been considered to have about an optimum of practicable efficiency.

Method of Counting.

The indirect method of estimating the true number of scintillations on the zinc sulphide screen, as used by Geiger and Werner, although ingenious, is certainly open to criticism. Their observations were made by counting simultaneously on the front and back faces of the screen. As different counts were obtained by the two observers, the legitimate assumption was made that observer I only sees a fraction λ_1, and observer II a fraction λ_2 of the true number of scintillations (N). Thus we have $N_I = \lambda_1 . N$, and $N_{II} = \lambda_2 . N$, and although these two fractions will, on the average, be fairly constant for the two observers, it is just as certain that the fraction for any one observer varies appreciably from day to day. A still greater element of uncertainty is introduced by their method of deriving N from these observations. The number of coincidences during a count is assumed to be $C = \lambda_2 . N_I = \lambda_2 . \lambda_1 . N$, or $C = \lambda_1 . N_{II} = \lambda_1 . \lambda_2 . N$, whence

$$N = \frac{N_I \cdot N_{II}}{C.}$$

Theoretically, this might be justifiable, if all the scintillations were of equal intensity, and if the dual registration of the scintillations were *instantaneous*; but in practice it is not. Geiger and Werner registered their counts on a chronograph

by electrical means, and using a tapping key each. Now a certain time lag occurs between the observation and the registration of a scintillation, which, as well as being different on the average for the two observers, will also depend on circumstances, such as fatigue, the frequency of scintillations, etc.; and what will happen when two or more scintillations occur simultaneously? Moreover, from our own observations, we know that each observer is slightly influenced by the tapping noise produced by the other observer during registration. It was for that reason that we, in our simultaneous counts, soon gave up the chronographic method of registration, and resorted to silent counting, the numbers being entered at definite periods by a third person.

From what has been said, it is clear that an appreciable latitude will be taken in estimating the number of coincidences, and almost unconsciously one will be inclined to regard as simultaneous, marks which lie approximately one over the other. The tendency will thus be to over-estimate the number of coincidences (C), and hence to under-estimate the value of N. This tendency will be greater, the greater the frequency of the scintillations, and would hardly be absent for the frequency of ca. 40 per minute, as used by Geiger and Werner.

It has been mentioned above that the factor λ would be expected to vary from time to time for each observer, and not necessarily in the same sense. In view of this, it is somewhat surprising to find (*loc. cit.* p. 196) the value of the factor for three different observers given to three places of decimals (viz. 0·884, 0·904, 0·917). These values have been derived from the total numbers of scintillations counted by each of the three observers working in pairs, and under different combinations. It would have been more interesting if a table had been constructed of the value of λ for each experimental series, and for each of the combinations of two observers. The accuracy to be expected in the determination of N would then have been more obvious, by consideration of the variation of λ for any one observer from one experimental series to the next. In consequence of such variations, the introduction of a third observer (and hence of a third value of λ) into the series of measurements has probably been to reduce the accuracy attainable, if the total number (ca. 30,000) of scintillations had been counted by two observers only.

General Remarks.

Geiger and Werner suggest that the difference between their results and ours might be due to the presence of small electrometer kicks of secondary origin in our experiments, and they draw attention to our paper (*Wien. Ber.* cxxvii. p. 463, 1918) on the photographic registration of alpha-particles to support their contention. We cannot allow this remark to pass unchallenged. In the first place, the numerous experimental series on which we based our final result were performed by the subjective method of counting; in this the average magnitude of an electrometer kick was 10 scale divisions, and throws differing from this order of magnitude were absent, except for an occasional small throw due to vibration caused by passing traffic, and never larger than about 0·5 division. The photographic method was found less reliable under our conditions than the subjective method, and was hence abandoned.

Nevertheless, it is important to point out that, given a sensitive string electrometer of very small period, and using an alpha-particle frequency of say 40 per minute, the photographic method would be ideal for such work. Unfortunately, such an electrometer was not available for our work, and the Elster-Geitel electrometer was found to be too tardy in its action to give accurate results. The string required from 1 to 1½ seconds to return to its zero line after a kick, and 0·13 to 0·2 second to reach its maximum deflexion. This introduced difficulties in counting a rapid succession of alpha-particles in the counter; for if two particles followed each other within 1/5 sec. they were registered as a double throw, but if the second one occurred as the string was returning to its stationary position, the momentum it imparted to the string was partially counter-balanced by the counter-momentum already possessed by the string. It is clear that variable throws were thus obtained under such conditions by the photographic method, some o them relatively small humps on the already deflected record. But it is quite unjustifiable to suggest that these throws were due to secondary causes. They rendered such counts uncertain, and for this reason we abandoned the photographic method.

In view of the result obtained by Geiger and Werner, it is interesting to recall that, by the photographic method, and taking double kicks as single ones, we found a result identical with that of these authors. However, by applying Marsden and Barratt's formula to take account of the

occasional rapid succession of particles, we were able to conclude "from these experiments, that 'Z' must be greater than $3{\cdot}6 \cdot 10^{10}$, and is probably not far removed from $3{\cdot}7 \cdot 10^{10}$." This is in good agreement with our visual counts by the subjective method of counting (viz. $3{\cdot}72 \cdot 10^{10}$), for which we can vouch a much higher degree of accuracy.

One of the most accurate methods of evaluating Z theoretically is by equating the kinetic energy of the alpha-particle and the recoil atom from radium to the thermal energy (in C.G.S. units) as obtained from accurate measurements on the heat production of radium. Using Geiger's value for the velocity of the alpha-particles from radium, and Hess's value for the hourly heat production from radium, we calculate for Z the value $3{\cdot}83 \cdot 10^{10}$. This value is about 3 per cent. higher than that we obtained experimentally, but more than 10 per cent. greater than the result of Geiger and Werner. These authors conclude from their measurements that there can "hardly be any further doubt that in the radioactive transformation of the radium atom, apart from the kinetic energy of the alpha-particle and of the recoil atom, additional energy is liberated in consequence of the readjustments taking place in the atomic nucleus." As there are six such transformations between Ra and RaD, Geiger's results seem to involve that 6 times 3 or 18 calories per hour are to be ascribed to this extra liberated energy resulting from transformation, or 13 per cent. of the total measured heat production per 1 gram of Ra (plus Em, A, B, and C). This appears to be an exorbitantly high figure, particularly in view of the fact that the beta- and gamma-rays from Ra (B + C) respectively contribute only 3·4 per cent. and 4·7 per cent. of the total heat produced by radium and short-lived decay products. That there is a probability of liberation of extra energy during transformation we do not deny, but in our opinion the magnitude of such energy is more likely to be in accord with our own experimental results than with those of Geiger and Werner.

Summary.

After careful examination of the whole of the experimental evidence, we are still of the opinion that the number of alpha-particles emitted per second by 1 gram of radium lies very near to $3{\cdot}72 \cdot 10^{10}$, the value we obtained in 1918. In the present paper we state our reasons for believing that the value ($3{\cdot}40 \cdot 10^{10}$) recently obtained by Geiger and Werner, using the scintillation method, is too low. In connexion

with Geiger and Werner's work, the following points have been discussed :—

(1) It seems probable that after fusing off and allowing their emanation containers to cool, an appreciable fraction of the emanation would be occluded in the platinum lead-in tube of the container, and hence a corresponding fraction of the alpha-particles would be ineffective.

(2) Although the scintillating power of the zinc-sulphide crystals used in making the screens was tested, it is doubtful whether their efficiency is retained after they have been mixed with a binding agent, used as adhesive.

(3) Geiger and Werner's method of determining the true number of scintillations occurring in a given interval of time may readily lead to an under-estimate of this number, owing to the uncertainty in evaluating exactly the number of coincidences for the two observers.

The criticism of our photographic records raised by these authors has been shown to be irrelevant, as our final determinations were not performed by the photographic method.

The University,
Graz.

The University,
Sheffield.

XXIII. *Reflexion of Plane Polarized Light by Etched Metals.* By Miss Olwen Jones, *B.Sc.**

[Plate V.]

§ 1. When plane polarized light is reflected normally from an etched metal surface, and the reflected light is analysed by a nicol prism, changes in the intensity of the light occur as the metal surface is rotated in its own plane. Under microscopic examination, changes in the individual crystals can be observed.

In the examination of the specimens, white light or the light of a mercury arc lamp was used—filters being sometimes employed. The light passed through an aperture in the side of the microscope tube and was reflected on to the metal surface by a glass slip. When plane polarized incident light was required, the polarizing nicol N_1 was placed with its shorter diagonal parallel to the glass slip. In what follows the horizontal direction in which the light first enters the illuminator will be referred to as E and W. The incident electric vibrations for plane polarized incident light were therefore N and S. The analysing nicol N_2 was attached to the eyepiece. When N_2 is placed so as to extinguish the light travelling up the microscope tube, if no change of

* Communicated by Mr. J. H. Shaxby.

polarization occurs on reflexion (*e. g.* normal reflexion at a polished surface), the nicols are virtually crossed and are referred to as such. Similarly they are virtually parallel when N_2 is rotated through 90° from the crossed position.

The metals examined included steels, brasses, aluminium-bronzes, and copper-nickel, also pure metals—copper, bismuth, nickel, and aluminium. They were polished and etched in the usual way. Nitric acid was most frequently used, and for copper alloys a solution of ferric chloride and hydrochloride acid. The reagents are referred to in the course of the paper.

§ 2. Very lightly etched specimens of steel were first examined exhibiting well differentiated pearlite and ferrite structure. On rotation of the stage under crossed nicols the ferrite remained uniformly dark, but the pearlite underwent changes passing through four maxima and four minima in a complete revolution. The pearlite showed the usual striated structure, and it was observed that the minima, which were in general complete extinctions, occurred when the striations were either E and W or N and S in the field, *i. e.* either in, or perpendicular to, the plane of polarization of the incident light.

Fig. 1.

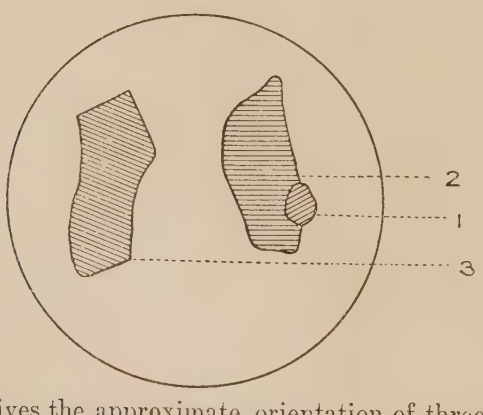


Fig. 1 gives the approximate orientation of three pearlite grains when the stage reading was 44°. The stage readings for the minima of three crystals were :—

(1)	17°	(striations horizontal)	106°	196°	283°
(2)	44°	(„ „)	133°	223°·5	322°
(3)	69°·5	(„ „)	160°	251°·5	339°

The minima therefore occurred at intervals of 90°.

Between the minima the crystals brightened up considerably and gave maxima when the striations were inclined at 45° on either side of the N and S line. The same specimen was examined with the nicols parallel. When the stage was rotated the ferrite remained uniformly bright, while the pearlite went through two maxima (striations N and S) and two minima (striations E and W).

A specimen of Muntz metal (30 per cent. Zn, 70 per cent. Cu) was lightly etched with nitric acid. It exhibited the usual duplex structure—the larger crystals of the α phase and the smaller complex laminated crystals of the mixed β and γ phase. When examined under high power, faint striations were visible in the second group of crystals but not with α crystals. On rotating the stage under crossed nicols the second group underwent the series of changes noted in the pearlite, while most of the α crystals remained uniformly dark. On etching more deeply with nitric acid, however, the α crystals went through the four maxima and four minima in a complete revolution, and on examination under high power showed faint striations. The striations in both sets of crystals were N–S or E–W in the minimum positions.

Pure copper was lightly etched with ammonia and examined. Individual crystals showed the four maxima and four minima. On etching more deeply with nitric acid and examining under high power, the striations in the crystals became distinct and the directions in the field of these at the maxima and minima were exactly as before; also on examining under parallel nicols, the maxima occurred when the striations were N–S and the minima when they were E–W, *i. e.* two maxima and two minima, as in the steel*.

Similar results were obtained in the case of bronze (10 per cent. Al). A specimen of bismuth lightly etched with nitric acid showed several crystals with well marked maxima and minima, the stage readings for a characteristic grain being

Minima	159°	247°	339°	67°
Maxima	205°	296°	26°	114°

* In several of the above cases it was at first thought that one pair of maxima (separated by 180°) was brighter than the other pair, *e. g.* with a certain bismuth crystal two maxima (at 303° and 123°) were much brighter than the other two (at 213° and 30°). The minima were not completely dark and their positions (70°, 173°, 251°, and 350) did not bisect the angles between the maxima. Similar effects were sometimes observed in other specimens. They were traced to inaccuracy in

A lightly etched brass examined under parallel nicols showed some crystals remaining uniformly bright and others with two maxima and two minima. Under crossed nicols the first set remained constantly dark, the second set again showing four maxima and four minima.

If light was allowed to fall directly on the crystal from the side (ordinary oblique illumination) and no nicols used, there were two maxima and two minima.

The readings for a single crystal under various types of illumination are :—

(a) *Under parallel nicols.*

Maxima ..... 231° 50° (not sharply defined).

Minima ..... Not sufficiently sharp to be determined with any degree of accuracy. Approximately half-way between the maxima.

(b) *Under crossed nicols.*

Maxima	180°	272°	0°	90°
Minima	227°	317°	46°	135°

(c) *Oblique incidence.*

Maxima ..... 225° 45°

Minima ..... Very little light reflected at any position between the maxima.

The photographs in Pl. V. are of aluminium bronze +·2 per cent. phosphorus (spec. 8).

In the first pair, *a* shows the appearance with unpolarized light, *b* with crossed nicols in one particular orientation.

The 2nd and 3rd pairs with crossed nicols illustrate the effect of rotating a specimen through 45° ; the maxima in *a* pass into the minima in *b*, and *vice versa*.

§ 3. The most interesting specimen examined (No. 8) was of composition Cu 94·8 per cent., Al 5 per cent., P 0·2 per cent. In addition to the maxima and minima it showed

---

the setting of N_2. For an exactly crossed position of N_2 (as tested by complete extinction of light reflected from a polished surface) the maxima were of the same intensity and separated from each other by 90° and from adjacent minima by 45°. If N_2 was then displaced very slightly to one side or the other one or other pair of maxima brightened up considerably, the other pair becoming correspondingly darker. In fact this affords quite a sensitive means of setting the nicols accurately in the crossed position.

very striking colour changes with the rotation of the stage or of N_2.

When very lightly etched with ferric chloride and hydrochloric acid and illuminated in the ordinary way without polarized light, the different crystals were only distinguished with difficulty and there were no signs of fine structure. The surface showed a large number of pits which under high power were seen to be partially flowed over by the surface film in the manner described by Sir G. Beilby.

On examination under crossed nicols the individual crystals were at once clearly differentiated, being of various degrees of brightness. The crystals were large and irregular and without the usual sharp corners. Each went through four very sharp maxima and minima in a complete rotation. With uncrossed nicols there were again two maxima and minima, these not being very well defined.

The following readings were taken for an individual crystal :—

(a) *Crossed nicols.*

Maxima	268°	354°	83°	173°
Minima	313°	43°	130°	223°

(b) *Uncrossed nicols.*

Maxima	315°	135°

As the stage was rotated under crossed nicols each crystal went through a definite colour series. The colours were much more marked if N_2 was shifted slightly from its crossed position.

If the stage was fixed so that a given crystal was at 45° from a minimum under crossed nicols and then N_2 was rotated slightly in, say, a clockwise direction, the reflected light became red. If N_2 was rotated slightly in an anticlockwise direction the light became green. The same colour changes occurred with the stage rotated through 180° from this position, while if the stage was turned through 90°, clockwise rotation of N_2 produced the green and anticlockwise rotation produced the red.

Similar colour changes were also observed in some specimens of 95 per cent. Cu, 5 per cent. Al, but the general effect was much less marked.

The origin of the colours is not at present clear. The brilliant effects noted in specimen 8 appear to depend a great deal on the depth of etching and the reagent used, and

are obtained with difficulty. An increase of the red or the green on rotation of N_2 at a maximum can be more frequently distinguished, although it is closely allied to the whole effect noted in 8.

The colouring is quite distinct from ordinary tarnishing, as tarnished specimens showing bright colour under ordinary illumination have been examined and do not show the colour sequences under crossed nicols, often appearing quite dark.

The effect apparently depends on the polarization of the light. Specimen 8 was subsequently etched more deeply with nitric acid and examined. The brilliant effects were no longer present but slight colour appeared in parts. These parts were often still covered with a thin surface film of the metal. Several of the crystals showed very regular striations under high power.

§ 4. The observation of the maxima and minima with crossed and uncrossed nicols led to the conclusion that the production of the maxima and minima might be due to two successive reflexions at the metal surface. The azimuths of the maxima and minima were constant over the whole crystal surface even when this was very large. Thus, a bar of aluminium, which was kindly sent by Prof. H. C. H. Carpenter, contained some crystals measuring approximately 1 in. or $1\frac{1}{2}$ in. across and the appearances were the same over each whole crystal. The phenomena are obviously therefore due to the internal structure of a crystal disclosed by etching.

It has already been shown by several workers that a single crystal of a metal consists in general of similarly oriented parts. In the Bakerian Lecture of 1899 on the "Crystalline Structure of Metals," by Sir J. A. Ewing and Professor Rosenhain, it is stated that:

> "Each grain is built up of similarly oriented parts but the orientation changes from grain to grain. Etching a polished surface developes a multitude of facets which have the same orientation over the surface of any one grain but different orientations in different grains" (Phil. Trans. vol.193, p. 356).

The behaviour of polarized light would point to the fact that the surfaces of some of the crystals are really *furrowed*, *i. e.*, there are two sets of parallel faces reflecting light and not only one as is the case in the strained crystals described in the above paper (p. 363).

This supposition is supported by the fact that with obliquely incident light the crystals in general appear

brightly illuminated at two positions (separated by 180°) in a complete revolution of the stage. Most of the large aluminium crystals which can be examined individually without the microscope also show two positions for the maximum reflexion of obliquely incident light. Measurements of the direction of maximum reflexion at the two maxima showed that the two sets of planes were inclined at 90° or a few degrees more or less. Since the etched crystals reflect normally incident light, either

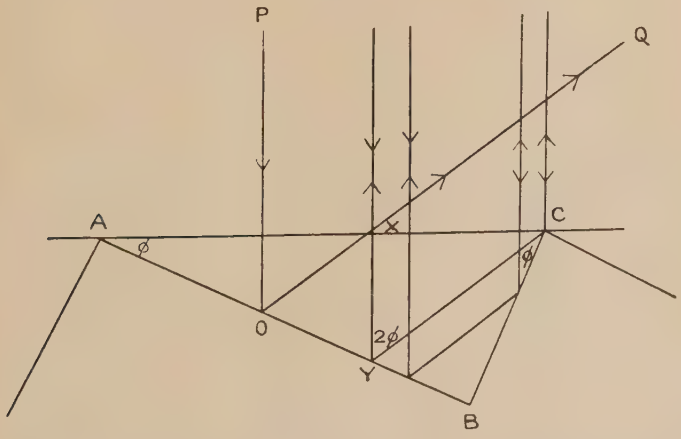
(*a*) the etching has not been deep enough completely to remove the surface film—giving a polished surface effect *e. g.* in the ferrite grains of lightly etched steel ; or

(*b*) the exposed furrows must have their faces inclined at an angle of approximately 90° since the light returns along the incident path.

A certain amount of deviation from 90° would be possible as the objective would gather in rays of small deviation.

§ 5. Suppose that a vertical section of an etched crystal is as in fig. 2. Vertical rays incident along AY will not be

Fig. 2.



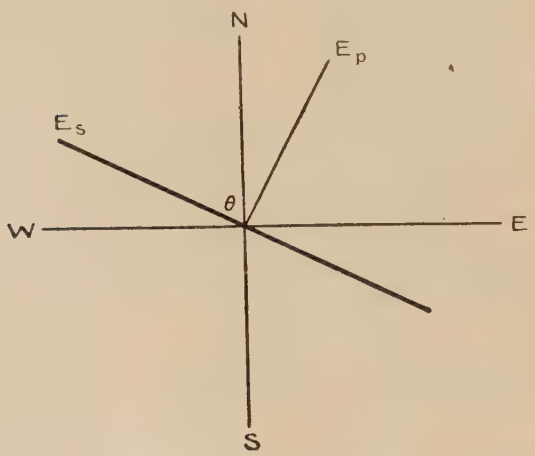
reflected at CB and so will be lost, *e. g.* ray POQ. The proportion of the surface therefore which is effective if only rays finally reflected vertically are considered, is

$$\frac{CX}{CA} = \frac{CA \sin\phi \sin 2\phi}{CA \cos\phi} = 2\sin^2\phi,$$

where ϕ is the inclination of the longer side to the horizontal.

Let the vertical incident light be plane polarized in a direction such that the incident vibrations (electric vectors) make an angle θ with the lines of intersection of the slant surfaces as in fig. 3, which represents the field of view. Let E_p and E_s (E being the amplitude of the incident light, fig. 3) be the components of amplitude of the incident light in and perpendicular to the plane of incidence on such a surface as AB (fig. 2) (*i. e.* polarized perpendicularly to and in the plane of incidence respectively).

Fig. 3.



Let ϕ = angle of incidence and R_p' R_s' be the corresponding amplitude components of the light reflected from AB.

This light then falls on the second surface BC at an angle ϕ'. Let R_p R_s be the final reflected amplitude components. The usual optical theory of reflexion by metallic surfaces gives

$$R_p = E_p \frac{\tan(\phi-\chi)}{\tan(\phi+\chi)} = E \sin\theta \frac{\tan(\phi-\chi)}{\tan(\phi+\chi)}$$

and

$$R_p = E \sin\theta \left\{\frac{\tan(\phi-\chi)}{\tan(\phi+\chi)}\right\} \left\{\frac{\tan(\phi'-\chi')}{\tan(\phi'+\chi')}\right\} = K_1 \sin\theta.$$

Similarly,

$$R_s = E \cos\theta \left\{\frac{\sin(\phi-\chi)}{\sin(\phi+\chi)}\right\} \left\{\frac{\sin(\phi'-\chi')}{\sin(\phi'+\chi')}\right\} = K_2 \cos\theta.$$

K_1, K_2 being constants as far as amplitudes are concerned,

since in a given crystal as the stage is rotated ϕ and ϕ' remain the same.

The resultant amplitude R along the E W line (*i. e.* the direction of vibrations transmitted by the analyser) is then given by resolving R_p R_s in that direction, *i. e.*

$$R = (K_1 - K_2) \sin\theta \cos\theta = \tfrac{1}{2}(K_1 - K_2) \sin 2\theta.$$

Hence the intensity is a maximum when

$$2\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2},$$

$$i.e. \qquad \theta = 45° \text{ or } 135°,$$

and no light will be transmitted when $\theta = 0°$ or $90°$.

Hence maxima occur four times in each revolution in accord with the observed phenomena.

With parallel nicols N_2 transmits light along the N S line. Therefore the resultant

$$= K_1 \sin^2\theta + K_2 \cos^2\theta$$

$$= K_2 - (K_2 - K_1) \sin^2\theta.$$

This gives maxima or minima at $\theta = \frac{n\pi}{2}$;

$$\text{at} \quad \theta = 0° \text{ or } 180°, \qquad R = K_2,$$

$$\text{at} \quad \theta = 90° \text{ or } 270°, \quad R = K_1.$$

$$\text{Here } K_2 > K_1.$$

Hence maxima occur when the join of the slant surfaces is parallel to the incident vibrations and minima when it is perpendicular thereto, *i. e.* 2 maxima and 2 minima in a complete rotation.

With a view to obtaining a comparison of effects, blocks of metal were set up to represent the "furrows," the angle between the faces being 90°. This was done with pairs of blocks of copper, bismuth, brass, nickel, and copper-nickel, and the furrows examined under the same conditions as the crystals.

In all cases they showed the same maxima and minima as the crystals, the directions of the furrows corresponding to the directions of the striations in the crystals. The effects were independent of the use of lenses in examination.

Further evidence that the cause of the polarized effects was to be found in the furrowed nature of the surface was

afforded by scratches which under crossed nicols showed bright maxima when at 45° on either side of the plane of incident polarization. A ruled metallic grating was also examined under the same conditions and showed very well marked maxima and minima, the minima again occurring when the lines were in or perpendicular to the plane of incident polarization and the maxima at 45° on either side.

In conclusion I wish to express my indebtedness to Mr. J. H. Shaxby for valuable help and advice throughout the course of the work, to Mr. C. Handford for assistance in the preparation of some specimens, and to Prof. A. A. Read and Prof. H. C. H. Carpenter for providing us with some of the specimens examined.

Summary and Conclusions.

Etched metal surfaces were illuminated by normally incident plane polarized light and the reflected light passed through a nicol prism.

If the analyser is placed so as to extinguish the light reflected from a polished part of the surface ("crossed" nicols) the etched parts show various degrees of brightness according to the orientation of the crystal. As the surface is rotated through 360° in its own plane each etched crystal exhibits 4 maxima and 4 minima. If the analyser is turned through 90° the crystals give 2 maxima and 2 minima.

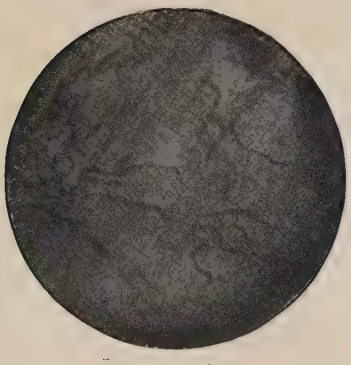
In some crystals showing the above effects it was possible to distinguish a fine striated structure of the surface under high magnification (1800 in some cases).

The view is put forward that the maxima and minima are due to the similar orientation of the parts making up any one crystal grain. This regular orientation is developed by etching into a ridged or furrowed structure of the surface, producing the striated appearance seen under high powers. Hence light incident normally on the surface of such an etched specimen is returned after two reflexions, one at each face of a furrow. The state of polarization of such light is shown to account for the maxima and minima.

Confirmatory observations were made by setting up large-scale models of such furrows by means of polished blocks and by examining scratches and the rulings of a metallic grating.

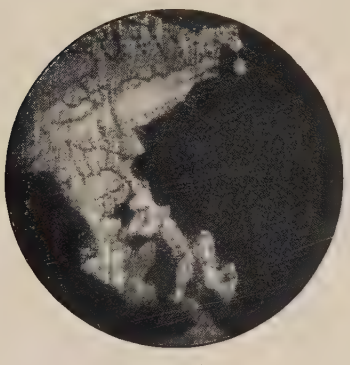
Viriamu Jones Physical Laboratory,
University College, Cardiff.

Fig. 1 *a*.



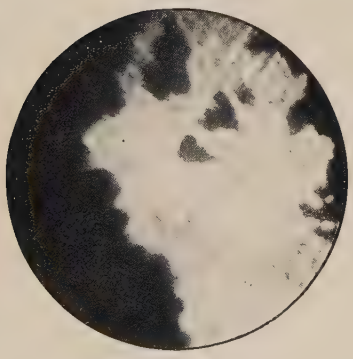
× 25.

Fig. 1 *b*.



× 25.

Fig. 2 *a*.



× 25.

Fig. 2 *b*.

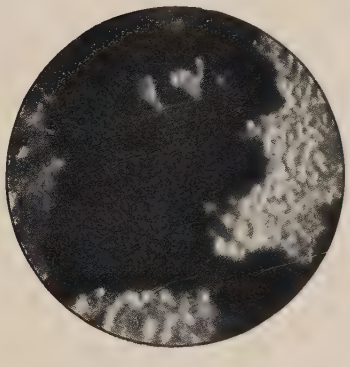


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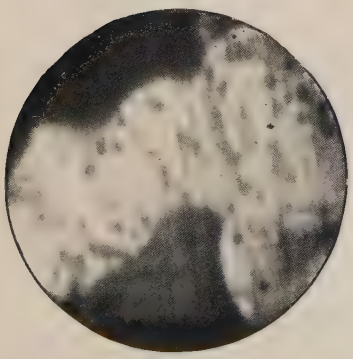
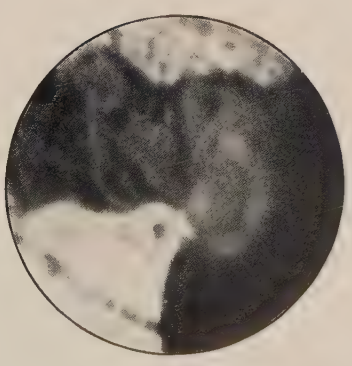
Fig. 3 *a*.



× 100.

Fig. 3 *b*.



× 100.

XXIV. *A Registering Micro-Photometer for Accurate Measurements.* *By* Prof. Manne Siegbahn, *D.Sc.**

FOR many purposes, especially in the spectroscopy of ordinary light and X-rays, a registering micro-photometer is needed. A photometric registering of a spectral plate in many cases shows a great deal more than can be seen directly on ocular viewing of the plate. Also an erroneous conclusion, caused by the physiological impress for instance through the contrast effect, may be avoided. There have also been constructed and used a variety of different micro-photometers, both of automatic registering type and also for subjective work, with this motive. Some years ago the author† built an automatic registering apparatus, where a fine pencil of light, after passing the photographic plate, fell on a thermopile, whose electromotive force was then registered with the help of a galvanometer and a drum. The drum and the photographic plate were moved through different gears driven from one and the same motor. But a registering in this way could not be used for more accurate measurements when it was wanted that a certain distance on the registering drum should correspond to a definite distance on the plate. A difference in the tension of the straps causes, of course, an alteration of the relation between the two distances.

Considerably better results were afforded by a later construction ‡, where a steel band connected the registered plate with the registering drum. In this apparatus the tension of the steel band was held constant during the registering by a load acting on the drum. Some inconvenience, which it is not necessary to mention now, caused me to construct a new model, which it was hoped would also be able to give still greater accurateness in the measurements. This apparatus, of which I will now give a short description, is very convenient in use, and enables measurements of line-distances to be made with a precision of about 0·002 mm.

From the three figures (figs. 1–3) the construction and working of the instrument may be seen. The plate to be photometered is placed horizontally on the slide B. This slide consists of two micrometrically adjustable parts, where the displacement of the photographic plate can be read directly to 1/100 of a millimetre and the thousands may be estimated. When the photometer is used for the registering

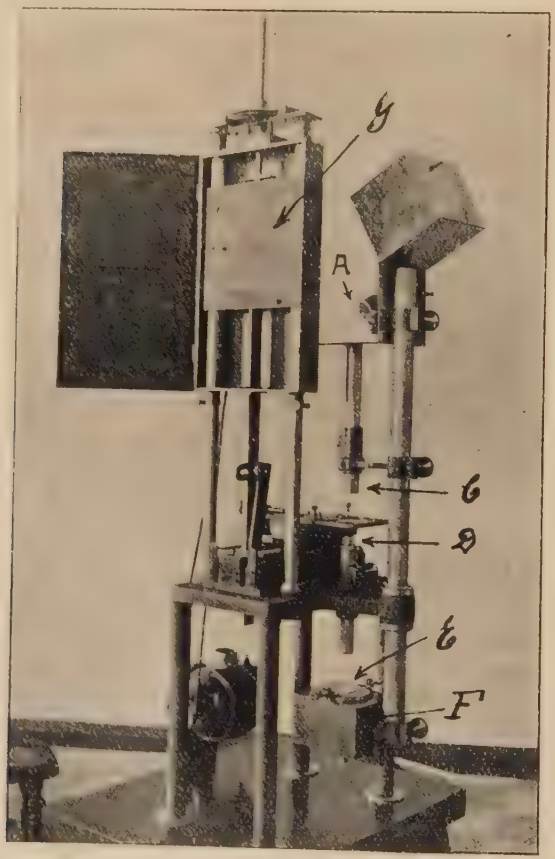
* Communicated by the Author.

† Phil. Mag. May 1914, p. 910; *Ann. d. Phys.* 1913, p. 689; Malmer, Diss. Lund, 1915.

‡ Described by Lindh, *Zs. für Phys.* vi. p. 303 (1921).

of plates, this arrangement is intended to enable different parts, say, of a spectral line to be set in for the registration. For subjective use of the photometer through observation of the deviation of the galvanometer, the micrometer screws may be used for measuring the distances for instance between the blackening tops of two lines.

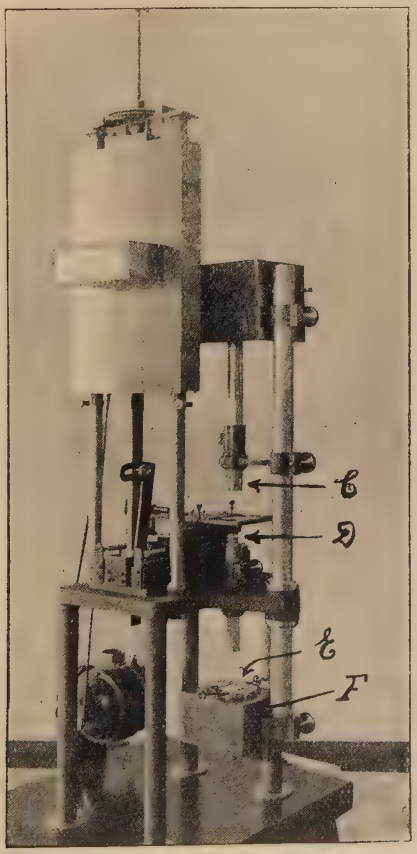
Fig. 1.



The optical part of the instrument consists of (1) an auto-lamp A (for 6–12 volts) with a straight spiral wire, (2) a condenser C, (3) an objective D, and (4) a micrometrically adjustable slit E. With the help of the condenser an image of the filament is thrown on the photographic plate. With the coated side of the plate turned down, an image of the film is formed on the slit. This operation is best done in the following way:—The condenser is lowered a little to get a greater portion

of the film illuminated. Then the objective is moved until a sharp image of the grains in the film is seen on a white paper gummed on the slit-backs. Also it is adjusted so that spectral lines are exactly parallel with the slit. After this is done the condenser is moved back to the place where a good image of the filament on the film is received. This last step may also, and with better results, be made through observing on the galvanometer when the maximum deviation is obtained.

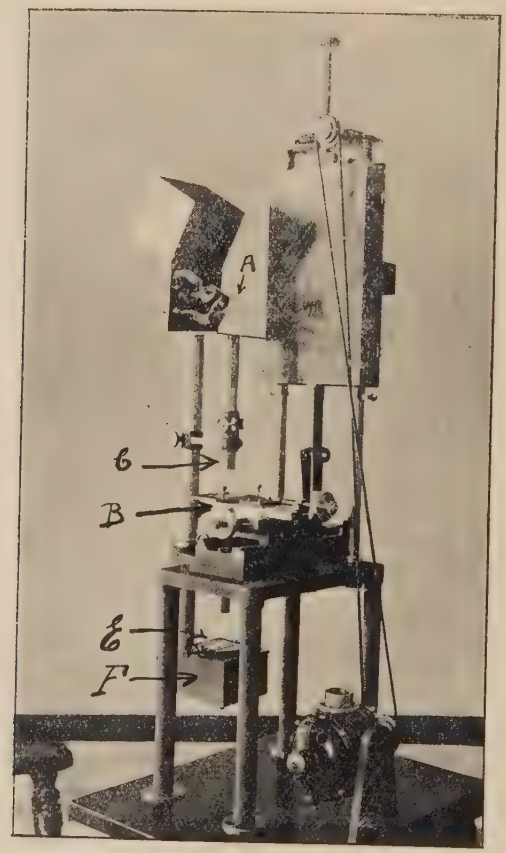
Fig. 2.



The slit-width, divided by the magnification of the objective, gives approximately the breadth of the part of the photographic plate which is registered. A magnification of about 10 times may be convenient, and with a slit-width of 0·2 mm. a band of about 0·02 mm. is cut out of the photographic plate. As to the length of the registered

band, this was in most cases here described about 1·5 mm., as the slit and the thermopile had a length of 15 mm. Naturally, a rectangular portion of these dimensions, that is $1{\cdot}5 \times 0{\cdot}02$ mm. of the photographic plate, shows irregularities in the blackening, owing to the accidental arrangement of the

Fig. 3.



grains in the film. On working with such small areas it is therefore necessary to make registrations on different parts of the spectral lines, and then to take a mean of many such curves. With four different registrations the accidental errors caused by the grains may be eliminated, as is shown in a paper by Mr. Larson and the author * where we used the apparatus to find the structure of some complicated X-ray

*Siegbahn and Larson, *Arkiv f. Mat. Astr. o. Fysik*, Stockholm, 1924.

spectral lines. As we then worked with plates having rather large grains, a smaller number of registrations may be sufficient on using plates with finer grains.

To get a stable zero-point of the galvanometer, it was necessary to have an effective shield for the thermopile F For this purpose the thermopile was placed in a heavy copper cylinder, and then this cylinder was enclosed airtight in a second box.

The most important part of the photometer is the arrangement by which the holder for the registering plate is connected with the slide for the registered plate. As the photometer was constructed with a view of enabling accurate measurements of distances on the *registrations*, it was necessary to use photographic plates instead of paper or film for registration of the galvanometer deflexions. These plates are fixed on a carrier G, which can be moved vertically and exactly on two steel bars of circular section. A third steel bar of rectangular cross-section is fixed to the carrier, and on the lower part of this one another similar steel bar is attached, as seen on fig. 1. The angle between these two steel bars may easily be varied as desired. When the carrier for the plate is going downwards, the wedge, formed by the last two steel bars, is pressed down between two agate cylinders, of which one is fixed to the bed and the other attached to the horizontal slide. Provided the two steel bars have *exactly straight* sides, a certain *vertical* displacement of the carrier G will correspond to a definite *horizontal* displacement of the movable slide carrying the plate.

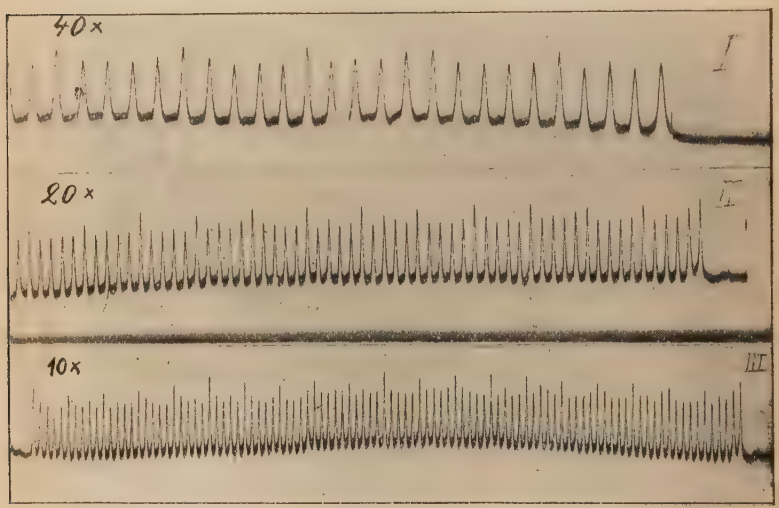
The arrangement to give the slow motion of the registering parts is seen on the upper side of the box for the vertical carrier. A heliacal screw acting on a nut brings down the rotation velocity of a small motor to a convenient amount. With the galvanometer and thermopile used in this apparatus the registration of a curve took about 5 minutes.

Some tests of the instruments may be given. The first of them was made in the following way:—An ocular scale as commonly used in microscopes, with a hundred divisions in 10 mm., was placed on the horizontal slide, and in the ordinary way photometered with three different wedge angles. These three angles were adapted to give a magnification (that is, the relation of the vertical to the horizontal motion) of about 10, 20, and 40. The three curves are seen in fig. 4. A measurement of the distances between the peaks gives values which do not differ from one another more than may be ascribed to dividing faults in the ocular scale. Eliminating these faults, the exactitude of the distances as measured on

the registered plate stays within 0·0025 mm. reduced to the distance on the scale.

It may be mentioned that the exactitude is determined principally by the precision with which the sides of the two steel bars are made. A still higher precision may certainly be attained if the steel bars are made with greater care than was the case in this instrument. For our purposes an accuracy of 0·0025 mm. was quite sufficient.

Fig. 4.



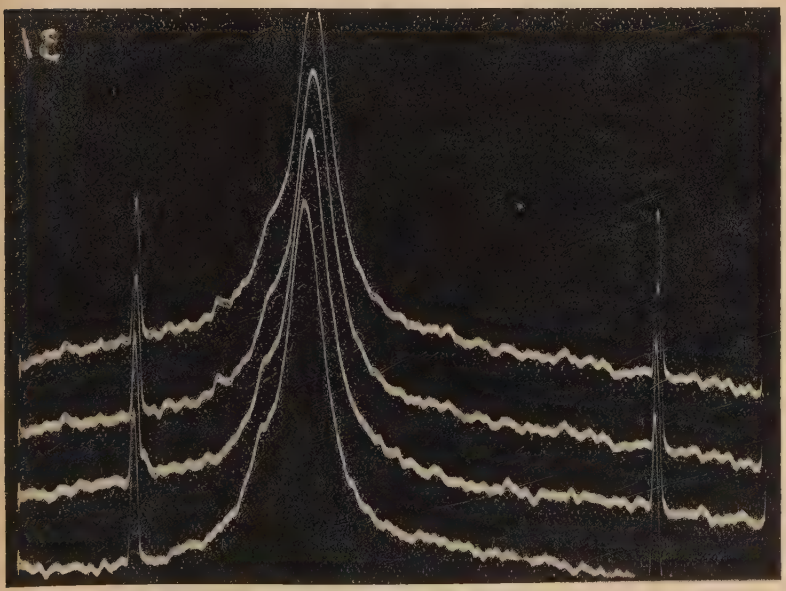
From the same figure it may be seen that the breadth of the scale lines is about 1/3 of the distance between the scale divisions. This corresponds to about 0·03 mm. as the width of the rectangular area cut out on the glass scale. As the magnification of the objective was about 10 times, this corresponds to a slit-width of 0·3 mm. Regarding this point, it is of interest to note that the image of the filament of the lamp is much broader. It shows that it is not at all necessary to use a special slit instead of the filament—as has sometimes been proposed—to get a good definition of the registered lines.

From the same figure it may also be concluded that the separation in this case was such that the two lines with a distance of 0·03 mm. will be wholly separated, and if the distance varies down to 0·015 mm., the curve will show a groove between the two peaks of the middle of the lines. There are no difficulties in getting a somewhat higher resolution than was given in this case, where the very short scale lines were used. With longer lines a finer slit could be used,

still giving the same total deviation. Also it is possible to give the lamp more current, which enables a further reduction of the slit-width without reducing the galvanometer deflexion.

As an illustration of the working of the registering apparatus, fig. 5 shows a plate taken from a rather complicated X-ray spectral line (Mo–Lα). Of the four curves the two uppermost are taken on the same place of the spectral plate. It will be seen that all the small deviations are exactly of the same shape in both these curves, showing that they are caused by small irregularities in the plate owing to the grains. The two other curves were taken after the spectral plate had been displaced a little parallel to the spectral lines.

Fig. 5.



As we said above, it is easy from some such curves, taken on different points of the spectral lines, to eliminate the accidental irregularities. The sharp peaks on both sides of the complicated spectral line are due to two lines scratched on the film with a sharp edge. They serve to determine the magnification when their distances on the spectral plate and on the registered plate are measured with a comparator.

It may be mentioned that an attempt was made, with the view of getting an automatic levelling of the irregularities caused by the grains in the film. As the spectral lines are rather long, an extension of the measured area in the

direction of the lines is possible. With the breadth still the same, the definition should be as good as before, but the irregularities strongly suppressed. This was done with the help of two cylindrical lenses shut in, the one between the condenser and the plate, and the other between the plate and the objective. But the result was not very good, as it is difficult to make the adjustments of the optical system accurate enough ; also the loss of intensity was rather great.

In the experiments described above, Mr. Erik Bäcklin has in a very skilful way assisted me. I wish to express here my hearty thanks for his valuable help.

Upsala, Physical Laboratory of the University,
March 17, 1924.

XXV. *On the Measurement of very small Changes of Capacity.* *By* Ross Gunn, *B.S.E.E., M.S., Instructor in Physics, Yale University*.*

THE ultra micrometer has proved of great value for the measurement of certain physical quantities, and while the methods of Whiddington (Phil. Mag. Nov. 1920) and Dowling (Phil. Mag. July 1923) give results, they leave much to be desired from the point of convenience, stability, and sensitivity. The writer would like to suggest two methods that are direct reading and very easy to use.

The method which will be described first is an extremely simple one, but is quite sensitive, giving a change of one microampere for a capacity change of 10^{-15} farad. By increasing the frequency and being careful with the design of the electrical circuits, this may be improved considerably.

The essential features of the circuit used are shown in fig. 1. The radio frequency oscillator A should be capable of delivering three or four watts at a frequency of about 1,500,000 cycles. The anode coil of the oscillator L_1 is coupled *loosely* to the coil L_2, which, together with the capacity to be measured, C_2, makes an oscillating circuit. This circuit, in which the losses should be kept as low as possible, is connected with another vacuum tube, as shown in the diagram. The grid biasing battery B_1 should have a potential of at least one-third that of the anode battery B_2. In general this will be more than sufficient to reduce the anode current to zero, when there is no impressed A.C. voltage in the circuit L_2C_2.

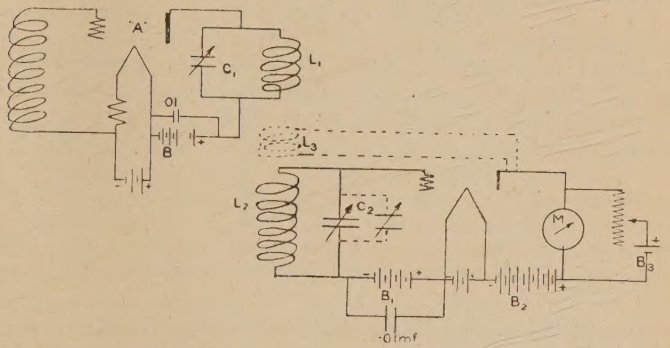
The sensitivity is somewhat greater when the impressed A.C. potential is sufficient to cause an anode current of about

* Communicated by the Author.

100 microamperes. The battery B_3 and resistance R are used to balance out this current if it is found desirable when using a very sensitive direct current microammeter or galvanometer.

To adjust the arrangement, place a shunt across the microammeter so that two or three milliamperes will give full scale deflexion. Tune the oscillator A by slowly changing C_1 until resonance is reached in the circuit L_2C_2, which condition will be indicated by a current through the meter.

Fig. 1.



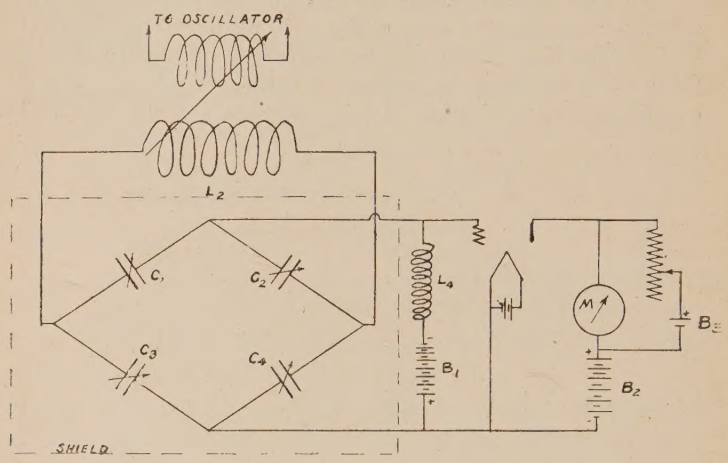
Loosen the coupling between L_1 and L_2 and remove the shunt from the meter. The most sensitive point will be found by adjusting the coupling so that a current of approximately 100 microamperes flows when the circuit L_2C_2 is adjusted to a point just off of resonance.

With this simple arrangement it is often possible to feed some of the anode energy back into the grid circuit with a considerable increase in sensitivity. One method of doing this is by means of the coil L_3, but in general it is hardly worth while, as the circuit is essentially an unstable one.

The second method (shown in fig. 2) is closely related to the one described above, for a capacity bridge replaces the single condenser formerly used. If the bridge and the coupling coil L_2 be so adjusted that they resonate approximately with the impressed frequency, and if one of the capacity arms of the bridge be changed, it is obvious that the potential impressed on the grid of the output vacuum tube will change, for two reasons: first, the usual variation caused by the changing of the impedance of the bridge-arms; and second, the variation due to tuning, as in the first case. It is then apparent that the most sensitive condition is realized when the bridge itself is not balanced. The diagram shows the arrangement of the various parts, and needs little explanation, as the details will occur immediately to the reader. It is

evident that the proper adjustment will require some skill, as there are two conditions to be satisfied simultaneously for maximum sensitivity.

Fig. 2.



It will be necessary to shield the bridge both from the surroundings and the associated tubes. With the arrangement used by the writer, the approach of a person within several feet produced a current change of a few microamperes. Shielding and proper arrangement of the parts eliminated this, and it was easily possible to obtain a sensitivity of one microampere for a capacity change of the order of 10^{-18} farad. This was accomplished by using a frequency of 1,500,000 cycles and capacities in the bridge of 125 micro-microfarads. Some attention must be given to keeping the distributed and stray capacities as low as possible.

XXVI. *Notices respecting New Books.*

The Calculus of Observations. A Treatise on Numerical Mathematics. By E. T. Whittaker, Sc.D., F.R.S., Professor of Mathematics in the University of Edinburgh, and G. Robinson, M.A., B.Sc., Lecturer in Mathematics in the University of Edinburgh. (Pp. xvi+395.) (London: Blackie & Son, 1924. Price 18*s.* net.)

THIS volume deals with the mathematical problems which arise in dealing with numerical data. Such problems are met with in many branches both of pure and of applied science, in astronomy, in meteorology, in physics and in engineering, for example. The statistician, the biometrician, and the actuary have also constantly to deal with them. In spite of the great practical importance of

the subject, it has been almost entirely neglected in the mathematical courses given at the various British Universities, with the exception of the University of Edinburgh, where Prof. Whittaker has given courses on the subject for the past ten years. These lectures form the basis of the present volume. All those who have to deal with masses of numerical data will be grateful to the authors for having expanded the substance of the lectures into book form. A wide range of subject matter is dealt with, including interpolation, central difference formulæ and their applications, the numerical solution of equations, numerical integration, error laws and the method of least squares, Fourier analysis, the smoothing of data, correlation and periodogram analysis.

Much of the material which is here gathered together was not readily accessible to the average student. Particular reference may be made to a valuable method for smoothing data, due to Prof. Whittaker himself, in which the degree to which fidelity to the original data may be sacrificed in order to obtain smoothness may be varied according to the nature of the problem.

The treatment throughout is lucid and rigorous and is illustrated by many numerical examples. The one criticism which we have to offer is that many of these examples deal with integral numbers or coefficients, which do not usually occur in practice. Thus the methods given, for example, for the solution of the normal equations in the method of least squares and for the determination of the weights of the several unknowns are not those which the practical computer would normally use.

The volume ably fills a notable gap in mathematical literature. It may be noted that the first four chapters are also published separately with the title 'A Short Course in Interpolation.'

Report on Radiation and the Quantum Theory. By Dr. J. H. Jeans. (2nd Edition.) (Fleetway Press. 7*s.* 6*d.* net.)

Perhaps no book on modern physics has been awaited more eagerly. Dr. Jeans always gives a fair and logical presentation of the case for and against any theory, and in this instance, as one of the pioneers whose work is too often passed over in current writings on spectra, is peculiarly fitted to do so. It is very essential that original writers in this country should not confine their writings to papers in scientific journals; for our share in the modern march of physics is being lost sight of, or ignored, even among those who, in a national sense, are the chief exponents at at present, even in our language. Perhaps this consideration should be emphasised no further, for there is no doubt in the mind of a reader of Dr. Jeans' Report as to the position of the subject at present. It is a final account of the theory as it exists to-day. But we must recommend readers to compare his Report with alleged accounts of the quantum theory which never mention the name of Jeans, or of other workers in this country, except in purely subsidiary ways.

Sommerfeld's brilliant generalization—and to a less extent even Bohr's angular momentum principle—were anticipated in this country by authors who receive little, if any, mention in current accounts. This detracts not at all from the brilliant nature of the work described; but Dr. Jeans is scrupulously fair, except perhaps to himself.

This may sound controversial, but not against the author. It is in some ways a protest, growing daily more insistent among workers, against the logical nature of present physical deductions. Physics will not advance by the persistency of advocates who either slur over or ignore difficulties, and refuse to even see phenomena which do not fit into a preconceived view of atomic structure and the relation of the quantum theory to it. The present Report is not open to this objection. From end to end, the author deals with every problem on its merits, and keeps an open mind in cases of doubt, but omitting nothing which is relevant, either in experimental research or in theoretical dynamics. In this respect, his account is unique, and of a value which it is difficult to estimate.

The reviewer is writing as one who believes in the fundamental validity of the quantum theory in certain respects—as one who believes in the views laid down by Bohr, but not the vague analogies of Bohr's followers, which are opposed to the logical nature of Prof. Bohr's own writings, which should be taken as a model. If we may take a particular case, pointed out already by others, Bohr's correspondence principle is a definite mathematical thing, whose application is precisely defined. But in the hands of his followers strange things are deduced from it, by analogies which entirely obliterate its real significance and value, and cannot, in fact, be logically connected with it. At this point we end a plea that the writings of Dr. Bohr be taken as the standard—Dr. Jeans has so taken them. The result is that his short discussion of the correspondence principle may disappoint many readers who wish to get ahead; but it is fair and final, and contains all that can really be deduced. Perhaps this review goes too much into detail on general principles. Yet such general principles are always present to those who really see the development of the quantum theory in a logical manner. Unfortunately, they do not include many of our experimental physicists. But they do include Prof. Bohr himself and the author of this Report.

The "adiabatic invariants" of Prof. P. Ehrenfest constitute, perhaps, the most beautiful sign of any hope of reconciling the quantum theory with the older dynamics. The author gives due space to this investigation.

We can only say, in conclusion, that the first fair picture of the present state of the quantum theory has been given—just as the first edition of this Report gave it so soon after the initial formulation of the quantum theory. The work before us is quite necessary to any library of modern physics.